Lecture 11: Numerical Steps and Stability

From the continuous to the discrete

Or...

The dangers of driving past your headlights
Agenda

- A quick look back at the 1rst Order Euler’s Method

- A discussion of the “Courant” Condition

- And how it reflects on Stella Modeling
What we’ve danced around

- The universe around us is a near-continuous system.

- Our models are highly discretized.
Why time steps matter

- Recalling our earlier adventures in learning about the numerical methods of forward-in-time integration (e.g., Euler and Runge-Kutta) we learned that reasonable time steps were critical to a sound simulation.

- In the case to the right, our discrete solution (green) is highly sensitive to time step.
Why time steps matter

- Increasing the discretization increases the accuracy but also increases the needs for storage and iteration time.
- One is left with the mandate to balance accuracy with efficiency (and even tractability)

\[
\frac{dy}{dt} = f(y_t, t)
\]

\[
y(t_0), t_0 \rightarrow y(t_0 + \Delta t), t_0 + \Delta t
\]

\[
y(t_1 + \Delta t), t_1 + \Delta t
\]
Why time steps matter

- One can, under the circumstances of a continuous system, select a higher-order RK scheme for example, but one is still left with these same challenges.

- And one may not always have the option of rotating schemes. E.g., RK schemes can fair poorly when unexpected “jerks” & “shocks” to the system are present.
Enter stability (driving within the view of the headlights)

- Driving past headlights is a trivial way to look at the too-long-a-time-step problem. However the analogy works to a degree.
  - Consider $y(t)$ to be the winding road.
  - Consider you to be the car.
  - Consider your $dy/dt$ at your “t” to be how you see the curve.

\[
\frac{dy}{dt} = f(y_t, t)
\]
Enter stability (driving within the view of the headlights)

- You have a certain amount of estimable tolerance of error
  - (noting that once you slide off the road, in this case, there may be no correcting factor to get you back on!)
  - This error should be seen as the maximum distance ($\Delta y$) you want to go, before your next “course check”

\[
\frac{dy}{dt} = f(y_t, t)
\]
Enter stability (driving within the view of the headlights)

- Getting that tolerance:
  - As yourself this:
    - What is a good estimate of how far I can go for any anticipated $\Delta y/\Delta t$?

\[
\frac{dy}{dt} = f(y_t, t)
\]

\[
f(y_t, t) = \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}
\]

\[
\frac{dy}{dt} = \frac{\Delta y}{\Delta t} \quad \text{(or so we'd like!)}
\]

\[
\Delta y = \frac{dy}{dt} \Delta t
\]
Enter stability (driving within the view of the headlights)

- Notice that for a system that starts with exponential decay, your first time step can be a whopper compared to subsequent steps.
- And in exponential growth, it can fall completely out of control since there are fewer constraints on your system.
- Some people have hedged this by applying a dynamic time step which increases with time.

\[
\frac{dy}{dt} = f(y_t, t)
\]
But can we get an Objective Criteria for Time Steps?

Actually yes!
Von Neuman Stability Analyses

- For certain systems which we can approximate as waves, this becomes “easy”
- Consider a wave (temperature, a water head anomaly, whatever) moving through a system as a speed “c”.
- Consider that a system is discretized in both space and time (a \(\Delta x\) and a \(\Delta t\))
- Our transport equation then becomes:

\[
\frac{\partial Q}{\partial t} + c \frac{\partial Q}{\partial x} = 0
\]

\[
\frac{\partial Q}{\partial t} = -c \frac{\partial Q}{\partial x}
\]
Von Neuman Stability Analyses

- By approximating the wave using Euler’s formula...

\[ Q_{j,n} = \xi^n e^{ik\Delta x} \]

- We can create a simple mathematical means to calculate a Euler’s method (no relation beyond the inventor, Leonhard Euler) for a condition for \( \Delta t \)

\[ \frac{|c| \Delta t}{\Delta x} \leq 1 \]
The Courant Condition

- This is the Courant-Friedrichs-Lewy condition
- Or Courant Condition for short.

\[ \frac{|c| \Delta t}{\Delta x} \leq 1 \]
Time to go to the canonical authority

“The Booke of Common Programming”

(1549 version shown, updated liturgies include C Pascal and F90/95)
Concretizing the Courant Eq.

- From Numerical Recipes:
  - Any “spatial” differencing scheme for a given spatial resolution has a finite “domain” of previous values in from the last time step by which a solution is extracted.
  - Let's create such a domain with one “spatial” dimension (x) and a time dimension (t).
  - This will be a discretized system with a uniform time step ($\Delta t$) and space step ($\Delta x$).
  - At our “forecast” time, the point for which we want to predict, will have no idea what’s going on at any of its neighboring points. But it WILL have access to the last time steps points both at its location and its neighboring values.
With Illumination in the Margins (not to mention good Fortran Code)

- Numerical Recipes has a comprehensive rundown on the various stability criteria for a number of adjective and diffusive transport representations.
- And more importantly they have this figure that explains things graphically!

![Diagrams showing stability criteria](image)

Figure 19.1.3. Courant condition for stability of a differencing scheme. The PDEs of an initial value problem imply that the value at a point depends on information within some domain of dependency to the past, shown here shaded. A differencing scheme has its own domain of dependency determined by the choice of points on one time slice (shown as connected solid dots) whose values are used in determining a new point (shown connected by dashed lines). A differencing scheme is Courant stable if the differencing domain of dependency is larger than that of the PDEs, as in (a), and unstable if the relationship is the reverse, as in (b).
With Illumination in the Margins (not to mention good Fortran Code)

Figure 19.1.3. Courant condition for stability of a differencing scheme. The PDEs of an initial value problem imply that the value at a point depends on information within some domain of dependency to the past, shown here shaded. A differencing scheme has its own domain of dependency determined by the choice of points on one time slice (shown as connected solid dots) whose values are used in determining a new point (shown connected by dashed lines). A differencing scheme is Courant stable if the differencing domain of dependency is larger than that of the PDEs, as in (a), and unstable if the relationship is the reverse, as in (b).
OK... It may help to draw it with color and a better explanation

- The green dot at is our forecast point.
- The colored dots are our previous and/or current time step points (with the pink ones being the ones to which we have access)

Where we “are” in space (or with respect to stocks)
OK... now that we’ve drawn it...

Let us now say that we have a velocity of “c” pushing our constituents through our system.
It may help to draw this *in Stella*.
Now lets send some material through our system.

- If we have a faster speed of the wave through the system, we increase the number of points that could influence our forecast from the previous time steps could out number those in your solving equations.

- At that speed, potential points contributing to our solution would fall in the grey triangle.

\[ |c| \Delta t > \Delta x \]
Now let’s send some material through our system – slowly

- Meanwhile, a slower wave would send the material at a slower speed through the system.

- This is where we want to be!

\[ |c| \Delta t < \Delta x \]
Now lets send some material through our system – slowly

- So lets take stock.
  - We should have a feel for what size of a “wave” we want to measure (our resolution), $\Delta x$.
  - We should know how fast these waves travel through our system (our phase speed), $c$.

- Given this, we calculate our $\Delta t$ using the Courant Condition:
  \[
  \frac{|c| \Delta t}{\Delta x} \leq 1
  \]
Living the Courant Lifestyle

So let's take stock.

- We should have a feel for what size of a "wave" we want to measure (our resolution), $\Delta x$
- The smallest "wave" we can represent is $2\Delta x$ in length from peak-to-peak
- (you can only see where it is and where it isn't)

\[ \frac{|c| \Delta t}{\Delta x} \leq 1 \]
Living the Courant Lifestyle

- So let's take some more stock.
  - We should also know how fast these waves travel through our system (our phase speed), \( c \). This will require foresight.

- Given this, we calculate our \( \Delta t \) using the Courant Condition:

\[
\frac{|c| \Delta t}{\Delta x} \leq 1
\]
Living the Courant Lifestyle
For the Discriminating Palate

So, if we were to demand a higher resolution... we should need to shrink the time step.

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Living the Courant Lifestyle
For the Jet Set

Likewise if the speed of our waves moving through the system, you'll need to shrink the time step...

\[ \frac{|c| \Delta t}{\Delta x} \leq 1 \]
Living the Courant Lifestyle
For the Jet Set

- Likewise if the speed of our waves moving through the system, you’ll need to shrink the time step...

- (and as with the jet-set, this one is living pretty close to the edge!)
Or to Summarize!

- The Higher the desired detail, the shorter the time step.
- The Faster the System Speed, the shorter the time step.

\[
|c| \Delta t \\
\Delta x \leq 1
\]
Or to “One-Sentence” it…

Select your time step so that you don’t “outrun (or suck dry) your stocks in any given time step!”
What does this have to do with Dueling Bunnies and Fudds?

- Everything: the principles are the same!
  - The Higher the desired detail (when moving in spatial contexts), the shorter the time step
  - Discretizing Steps when emulating a converter
  - The Faster the System Speed, the shorter the time step
  - Shorter Life spans, Faster Birthrates.