SECTION 12. Moment-Curvature Relationship before Flexural Cracking

In this section, we focus on the response of an uncracked reinforced concrete section to applied moment. A reinforced concrete beam in a structure may have cracks caused by shrinkage and temperature effects even if it has not been subjected to moments high enough to cause flexural cracking. This does not mean we should ignore the behavior of the uncracked section because there are uncracked regions in a cracked beam and the overall response of the beam in flexure depends on the responses of the uncracked as well as the cracked sections.

To develop a simple relationship between moment and curvature for a rectangular section (Fig. 12-1a) of a straight beam, we make two pivotal assumptions:

1. Distribution of unit strain over the section is linear (Fig. 12-1b).
2. Unit stress changes linearly with unit strain (Fig 12-1c)

It is relevant to note that both assumptions have their roots in observation. Short of explaining it in terms of molecular movement, strain linearity is justified by observation. The stiffness of the concrete (Young’s modulus, $E_c$) is also based on observation.

Considering the natural scatter expected in the values of Young’s modulus (as well as assuming it to be the same in tension and in compression) it is counterproductive to consider the effect of the reinforcement on flexural response.¹

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¹ Even though the presence of tensile reinforcement may appear to increase the cracking moment, it decreases it because shrinkage of the concrete is restrained by the reinforcement.
Because we have assumed a constant modulus in tension and compression, we recognize that the neutral axis (position with zero applied stress) will be at mid-height of the rectangular section. If the stress in the extreme fiber in compression is $f_c$, the stress at a distance $y_n$ from the neutral axis is

$$f(y_n) = \frac{y}{\frac{1}{2}h} f_c$$

Eq. 12-1

This unit stress acts on a differential of area (Figure 12-2):

$$dA = b \cdot dy$$

Eq. 12-2

and is associated with a differential of force:

$$dF = f(y_n)dA$$

Eq. 12-3

The resultant of compressive stresses is:

$$C = \int_{0}^{\frac{1}{2}h} f(y_n)dA = \frac{1}{2} f_c b \frac{h}{2}$$

Eq. 12-4

The product $f(y_n)dA$ can also be obtained by treating it as a differential of volume (Figure 12-2). In that case the magnitude of the force $C$ is the volume of the wedge shaded in red in Figure 12-2.

Because the beam is not subjected to axial load, $C$ must equal the resultant of tensile stresses $T$:

$$C = T = \frac{1}{4} b \cdot h \cdot f_c$$

Eq. 12-5

The lines of action of these resultants pass through the centroids of the volumes shown in Figure 12-2.

The distance between the two centroids is:
Equation 12-7 relates the maximum stress in the section to the applied moment. Its form is a consequence of the assumed distribution of unit stress. Unit stress distributions with different shapes lead to different results. The quantity \( \frac{bh^2}{6} \) is called the section modulus \( S \) and is equal to the ratio of moment of inertia of the full section divided by the distance from the neutral axis to the point where stress is maximum (in this case, \( h/2 \)). The section modulus is a function of the shape of the cross section.

We compute the maximum compressive strain as

\[
\varepsilon_{\text{max}} = \frac{f_c}{E_c} = \frac{M}{E_cS} \quad \text{Eq. 12-8}
\]

and unit curvature as

\[
\phi = \frac{\varepsilon_c}{c} = \frac{M}{E_cI_g} \quad \text{Eq. 12-9}
\]

where \( I_g = \frac{1}{12} \cdot bh^3 \) and \( c = h/2 \).

The boundary between Stages I and II is given by the moment at cracking \( M_{cr} \). For normal-weight concrete, a reasonable lower bound to the moment at cracking can be expressed as

\[
M_{cr} = S \cdot f_r = \frac{1}{6} \cdot b \cdot h^2 \cdot f_r = b \cdot h^2 \cdot \sqrt{f'_c} \quad \text{Eq. 12-10}
\]

Here \( f_r \) is modulus of rupture and is assumed to be \( f_r = 6\sqrt{f'_c} \) (Sec. 5).

**Essentials:**

Before cracking—in Stage I—the relationship between curvature and bending moment is:

\[
\phi = \frac{\varepsilon_c}{c} = \frac{M}{E_cI_g}
\]

Response remains in Stage I as long as the applied moment does not exceed:

\[
M_{cr} = S \cdot f_r
\]

For beams with rectangular cross sections, a reasonable lower bound to the moment at cracking can be expressed as:

\[
M_{cr} = S \cdot f_r = b \cdot h^2 \cdot \sqrt{f'_c}
\]
**Example**

For the reinforced concrete beam shown, and for the following parameters, compute the magnitude of the uniformly distributed load $w$ expected to cause cracking. Ignore self weight.

- $f'_c = 5000$ psi
- $L = 18$ ft
- $h = 18$ in.

**Solution**

The maximum “applied” moment is located at midspan and is equal to:

$$M_a = \frac{wL^2}{8}$$

The moment of inertia of the full section, corrected to three significant figures, is,

$$I_g = \frac{9 \text{ in.} \cdot (18 \text{ in.})^3}{12} = 4370 \text{ in}^4$$

The section modulus is:

$$S = \frac{I_g}{h/2} = \frac{4370 \text{ in}^4}{9 \text{ in.}} = 486 \text{ in}^3.$$  

Taking the modulus of rupture as $f_r = 6 \sqrt{5000}$ psi = 420 psi

We obtain $M_r = S f_r = 486 \text{ in}^3 \times 420 \text{ psi} \times (1 \text{ kip} / 1000 \text{ lbf}) \times (1 \text{ ft} / 12 \text{ in.}) = 17 \text{ kip-ft}$

Cracking is expected to occur when the applied moment reaches the moment at cracking ($M_a = M_c$). Solving for $w$:

$$w = 8 \frac{M_c}{L^2} = 8 \times 17 \text{ kip-ft} / (18 \text{ ft})^2 = 0.42 \text{ kip/ft}.$$  

**Exercise**

For the reinforced concrete beam shown, and for the following parameters, estimate the load expected to cause cracking. (Keep in mind that $S$ in the expression $M_c = S f_r$ is a function of the shape of the cross section and is equal to the ratio of moment of inertia of the full section divided by distance from the neutral axis to the point where stress is maximum.)

- $f'_c = 5000$ psi
- $L = 18$ ft