Agenda

Factory Physics
Chapter 2: Inventory Control From EOQ to ROP
Web Resources
http://sdmines.sdsmt.edu/sdsmt/directory/courses/2009fa/tm663M021-099
Really encourages you to use the search tool at the main sight.
I found a pdf printer. I hope it works for you.

Tentative Schedule

<table>
<thead>
<tr>
<th>Chapters Assigned</th>
<th>Chapters Assigned</th>
</tr>
</thead>
<tbody>
<tr>
<td>8/31/2009 0,1</td>
<td>11/23/2009 Exam 2</td>
</tr>
<tr>
<td>9/21/2009 2, 3</td>
<td>12/14/2009 Final</td>
</tr>
<tr>
<td>9/28/2009 4, 5</td>
<td>18, 19 Not covered We may rearrange a bit. We could skip chapter 11 &amp;12 and do 18&amp;19</td>
</tr>
<tr>
<td>10/5/2009 6, 7</td>
<td>10/12/2009 Holiday</td>
</tr>
<tr>
<td>10/19/2009 Exam 1</td>
<td>10/26/2009 8,9</td>
</tr>
<tr>
<td>11/2/2009 9,10</td>
<td>11/9/2009 11, 12</td>
</tr>
</tbody>
</table>
The EOQ Model

To a pessimist, the glass is half empty.  
To an optimist, it is half full.  

– Anonymous

EOQ History

• Introduced in 1913 by Ford W. Harris, “How Many Parts to Make at Once”

• Interest on capital tied up in wages, material and overhead sets a maximum limit to the quantity of parts which can be profitably manufactured at one time; “set-up” costs on the job fix the minimum. Experience has shown one manager a way to determine the economical size of lots.

• Early application of mathematical modeling to Scientific Management
MedEquip Example

- Small manufacturer of medical diagnostic equipment.
- Purchases standard steel “racks” into which components are mounted.
- Metal working shop can produce (and sell) racks more cheaply if they are produced in batches due to wasted time setting up shop.
- MedEquip doesn’t want to tie up too much precious capital in inventory.

- **Question**: how many racks should MedEquip order at once?

EOQ Modeling Assumptions

1. *Production is instantaneous* – there is no capacity constraint and the entire lot is produced simultaneously.
2. *Delivery is immediate* – there is no time lag between production and availability to satisfy demand.
3. *Demand is deterministic* – there is no uncertainty about the quantity or timing of demand.
4. *Demand is constant over time* – in fact, it can be represented as a straight line, so that if annual demand is 365 units this translates into a daily demand of one unit.
5. *A production run incurs a fixed setup cost* – regardless of the size of the lot or the status of the factory, the setup cost is constant.
6. *Products can be analyzed singly* – either there is only a single product or conditions exist that ensure separability of products.
**Notation**

- $D$: demand rate (units per year)
- $c$: unit production cost, not counting setup or inventory costs (dollars per unit)
- $A$: fixed or setup cost to place an order (dollars)
- $h$: holding cost (dollars per year); if the holding cost is consists entirely of interest on money tied up in inventory, then $h = ic$ where $i$ is an annual interest rate.
- $Q$: the unknown size of the order or lot size \( \leftarrow \text{ decision variable} \)

**Inventory vs Time in EOQ Model**

![Inventory vs Time in EOQ Model](image-url)
Costs

**Holding Cost:**
- average inventory = \( \frac{Q}{2} \)
- annual holding cost = \( \frac{hQ}{2} \)
- unit holding cost = \( \frac{hQ}{2D} \)

**Setup Costs:** \( A \) per lot, so unit setup cost = \( \frac{A}{Q} \)

**Production Cost:** \( c \) per unit

**Cost Function:**
\[
Y(Q) = \frac{hQ}{2D} + \frac{A}{Q} + c
\]

---

**MedEquip Example Costs**

- \( D = 1000 \) racks per year
- \( c = \$250 \)
- \( A = \$500 \) (estimated from supplier’s pricing)
- \( h = (0.1)(\$250) + \$10 = \$35 \) per unit per year
Costs in EOQ Model

Economic Order Quantity

\[
\frac{dY(Q)}{dQ} = \frac{h}{2D} - \frac{A}{Q^2} = 0
\]

**EOQ Square Root Formula**

\[ Q^* = \sqrt{\frac{2AD}{h}} \]

**MedEquip Solution**

\[ Q^* = \sqrt{\frac{2(500)(1000)}{35}} = 169 \]
EOQ Modeling Assumptions

1. **Production is instantaneous** — there is no capacity constraint and the entire lot is produced simultaneously.

2. **Delivery is immediate** — there is no time lag between production and availability to satisfy demand.

3. **Demand is deterministic** — there is no uncertainty about the quantity or timing of demand.

4. **Demand is constant over time** — in fact, it can be represented as a straight line, so that if annual demand is 365 units this translates into a daily demand of one unit.

5. **A production run incurs a fixed setup cost** — regardless of the size of the lot or the status of the factory, the setup cost is constant.

6. **Products can be analyzed singly** — either there is only a single product or conditions exist that ensure separability of products.

---

Notation — EPL Model

- **D** demand rate (units per year)
- **P** production rate (units per year), where \( P > D \)
- **c** unit production cost, not counting setup or inventory costs (dollars per unit)
- **A** fixed or setup cost to place an order (dollars)
- **h** holding cost (dollars per year); if the holding cost is consists entirely of interest on money tied up in inventory, then \( h = ic \) where \( i \) is an annual interest rate.
- **Q** the unknown size of the production lot size \( \Rightarrow \text{decision variable} \)
Inventory vs Time in EPL Model

Production run of $Q$ takes $Q/P$ time units

$\frac{(P-D)(Q/P)}{2}$

Solution to EPL Model

Annual Cost Function:

$$Y(Q) = \frac{AD}{Q} + \frac{h(1 - D/P)Q}{2} + Dc$$

setup holding production

Solution (by taking derivative and setting equal to zero):

$$Q^* = \sqrt{\frac{2AD}{h(1 - P/D)}}$$

• tends to EOQ as $P \to \infty$
• otherwise larger than EOQ because replenishment takes longer
The Key Insight of EOQ

There is a tradeoff between lot size and inventory

Order Frequency:

\[ F = \frac{D}{Q} \]

Inventory Investment:

\[ I = cQ \frac{2}{2} = \frac{cD}{2F} \]

EOQ Tradeoff Curve
Sensitivity of EOQ Model to Quantity

Optimal Unit Cost:

\[ Y^* = Y(Q^*) = \frac{hQ'}{2D} + \frac{A}{Q^*} \]

We neglect unit cost, \( c \), since it does not affect \( Q^* \)

\[ = \frac{h\sqrt{2AD}/h}{2D} + \frac{A}{\sqrt{2AD}/h} \]

\[ = \frac{2A}{\sqrt{2AD}/h} \]

Optimal Annual Cost: Multiply \( Y^* \) by \( D \) and simplify,

Annual Cost = \( \sqrt{2ADh} \)

Sensitivity of EOQ Model to Quantity (cont.)

Annual Cost from Using \( Q' \):

\[ Y(Q') = \frac{hQ'}{2} + \frac{AD}{Q'} \]

Ratio:

\[ \frac{\text{Cost}(Q')}{\text{Cost}(Q^*)} = \frac{Y(Q')}{Y(Q^*)} = \frac{hQ'/2 + AD/Q'}{\sqrt{2ADh}} = \frac{1}{2} \left( \frac{Q'}{Q^*} + \frac{Q^*}{Q'} \right) \]

Example: If \( Q' = 2Q^* \), then the ratio of the actual to optimal cost is

\[ (1/2)[2 + (1/2)] = 1.25 \]
Sensitivity of EOQ Model to Order Interval

Order Interval: Let $T$ represent time (in years) between orders (production runs)

$$T = \frac{Q}{D}$$

Optimal Order Interval:

$$T^* = \frac{Q^*}{D} = \frac{2AD}{h} = \frac{2A}{\sqrt{hD}}$$

Sensitivity of EOQ Model to Order Interval (cont.)

Ratio of Actual to Optimal Costs: If we use $T'$ instead of $T^*$

annual cost under $T' = \frac{1}{2} \left[ \frac{T'}{T^*} + \frac{T^*}{T'} \right]$

Powers-of-Two Order Intervals: The optimal order interval, $T^*$ must lie within a multiplicative factor of $\sqrt{2}$ of a “power-of-two.” Hence, the maximum error from using the best power-of-two is

$$\frac{1}{2} \left[ \sqrt{2} + \frac{1}{\sqrt{2}} \right] = 1.06$$
The “Root-Two” Interval

\[ \begin{align*}
2^m & \quad T_1^* & \quad 2^m \sqrt{2} & \quad T_2^* & \quad 2^{m+1} \\
\text{divide by} & \quad \text{less than} & \quad \sqrt{2} \text{ to get} & \quad \text{to} \ 2^m & \quad \text{multiply by} & \quad \text{less than} & \quad \sqrt{2} \text{ to get} & \quad \text{to} \ 2^{m+1}
\end{align*} \]

Medequip Example

**Optimum:** \( Q^* = 169 \), so \( T^* = Q^*/D = 169/1000 \) years = 62 days

\[
Y(Q^*) = \frac{hQ^*}{2} + \frac{AD}{Q^*} = \frac{35(169)}{2} + \frac{500(1000)}{169} = \$5,916
\]

**Round to Nearest Power-of-Two:** 62 is between 32 and 64, but since \( 32\sqrt{2} = 45.25 \), it is “closest” to 64. So, round to \( T' = 64 \) days or \( Q' = T'D = (64/365)1000 = 175 \).

\[
Y(Q') = \frac{hQ'}{2} + \frac{AD}{Q'} = \frac{35(175)}{2} + \frac{500(1000)}{175} = \$5,920
\]

*Only 0.07% error because we were lucky and happened to be close to a power-of-two. But we can’t do worse than 6%.*
Powers-of-Two Order Intervals

<table>
<thead>
<tr>
<th>Order Interval</th>
<th>Week</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 = 2^0$</td>
<td>0</td>
</tr>
<tr>
<td>$2 = 2^1$</td>
<td></td>
</tr>
<tr>
<td>$4 = 2^2$</td>
<td></td>
</tr>
<tr>
<td>$8 = 2^3$</td>
<td></td>
</tr>
</tbody>
</table>

EOQ Takeaways

- Batching causes inventory (i.e., larger lot sizes translate into more stock).

- Under specific modeling assumptions the lot size that optimally balances holding and setup costs is given by the square root formula:

  $$Q^* = \sqrt{\frac{2AD}{h}}$$

- Total cost is relatively insensitive to lot size (so rounding for other reasons, like coordinating shipping, may be attractive).
The Wagner-Whitin Model

Change is not made without inconvenience, even from worse to better.

– Robert Hooker

EOQ Assumptions

1. Instantaneous production.

2. Immediate delivery.

3. Deterministic demand.

4. Constant demand. \( \Leftarrow \) WW model relaxes this one

5. Known fixed setup costs.

Dynamic Lot Sizing Notation

\( t \) a period (e.g., day, week, month); we will consider \( t = 1, \ldots, T \), where \( T \) represents the planning horizon.

\( D_t \) demand in period \( t \) (in units)

\( c_t \) unit production cost (in dollars per unit), not counting setup or inventory costs in period \( t \)

\( A_t \) fixed or setup cost (in dollars) to place an order in period \( t \)

\( h_t \) holding cost (in dollars) to carry a unit of inventory from period \( t \) to period \( t+1 \)

\( Q_t \) the unknown size of the order or lot size in period \( t \) \( \downarrow \) decision variables

Wagner-Whitin Example

Data

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_t )</td>
<td>20</td>
<td>50</td>
<td>10</td>
<td>50</td>
<td>50</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>( c_t )</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( A_t )</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>( h_t )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Lot-for-Lot Solution

<table>
<thead>
<tr>
<th>( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_t )</td>
<td>20</td>
<td>50</td>
<td>10</td>
<td>50</td>
<td>50</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>20</td>
<td>30</td>
<td>300</td>
</tr>
<tr>
<td>( Q_t )</td>
<td>20</td>
<td>50</td>
<td>10</td>
<td>50</td>
<td>50</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>20</td>
<td>30</td>
<td>300</td>
</tr>
<tr>
<td>( I_t )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup cost</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>1000</td>
</tr>
<tr>
<td>Holding cost</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total cost</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>1000</td>
</tr>
</tbody>
</table>
Wagner-Whitin Example (cont.)

Fixed Order Quantity Solution

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dt</td>
<td>20</td>
<td>50</td>
<td>10</td>
<td>50</td>
<td>50</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>20</td>
<td>30</td>
<td>300</td>
</tr>
<tr>
<td>Qt</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>It</td>
<td>80</td>
<td>30</td>
<td>20</td>
<td>70</td>
<td>20</td>
<td>10</td>
<td>90</td>
<td>50</td>
<td>30</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>Setup cost</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>Holding cost</td>
<td>80</td>
<td>30</td>
<td>20</td>
<td>70</td>
<td>20</td>
<td>10</td>
<td>90</td>
<td>50</td>
<td>30</td>
<td>0</td>
<td>400</td>
</tr>
<tr>
<td>Total cost</td>
<td>180</td>
<td>30</td>
<td>20</td>
<td>170</td>
<td>20</td>
<td>10</td>
<td>190</td>
<td>50</td>
<td>30</td>
<td>0</td>
<td>700</td>
</tr>
</tbody>
</table>

Wagner-Whitin Property

Under an optimal lot-sizing policy either the inventory carried to period t+1 from a previous period will be zero or the production quantity in period t+1 will be zero.
Basic Idea of Wagner-Whitin Algorithm

By WW Property I, either $Q_t=0$ or $Q_t=D_1+...+D_k$ for some $k$. If

$$j_k^* = \text{last period of production in a } k\text{ period problem}$$

then we will produce exactly $D_1+...+D_T$ in period $j_k^*$.

We can then consider periods $1, \ldots, j_k^*-1$ as if they are an independent $j_k^*-1$ period problem.

----

Wagner-Whitin Example

**Step 1:** Obviously, just satisfy $D_1$ (note we are neglecting production cost, since it is fixed).

$$Z_1^* = A_1 = 100$$

$$j_1^* = 1$$

**Step 2:** Two choices, either $j_2^* = 1$ or $j_2^* = 2$.

$$Z_2^* = \min\left\{ A_1 + h_1D_2, \text{produce in 1} \right\}$$

$$= \min\left\{ 100 + 1(50) = 150 \right\}$$

$$= 150$$

$$j_2^* = 1$$
Wagner-Whitin Example (cont.)

**Step 3:** Three choices, \( j_3^* = 1, 2, 3 \).

\[
Z'_3 = \min \left\{ \begin{array}{l}
A_i + h_iD_2 + (h_i + h_2)D_3, \quad \text{produce in 1} \\
Z'_1 + A_i + h_2D_1, \quad \text{produce in 2} \\
Z'_2 + A_i, \quad \text{produce in 3}
\end{array} \right.
\]

\[
= \min \left\{ \begin{array}{l}
100 + 1(50) + (1 + 1)10 = 170 \\
100 + 100 + (1)10 = 210 \\
150 + 100 = 250
\end{array} \right.
\]

\( j_3^* = 1 \)

Wagner-Whitin Example (cont.)

**Step 4:** Four choices, \( j_4^* = 1, 2, 3, 4 \).

\[
Z'_4 = \min \left\{ \begin{array}{l}
A_i + h_iD_2 + (h_i + h_2)D_3 + (h_i + h_2 + h_3)D_4, \quad \text{produce in 1} \\
Z'_1 + A_i + h_2D_1 + (h_2 + h_3)D_4, \quad \text{produce in 2} \\
Z'_2 + A_i + h_2D_1, \quad \text{produce in 3} \\
Z'_3 + A_i, \quad \text{produce in 4}
\end{array} \right.
\]

\[
= \min \left\{ \begin{array}{l}
100 + 1(50) + (1 + 1)10 + (1 + 1)50 = 320 \\
100 + 100 + (1)10 + (1 + 1)50 = 310 \\
150 + 100 + (1)50 = 300 \\
170 + 100 = 270
\end{array} \right.
\]

\( j_4^* = 4 \)
Planning Horizon Property

If \( j^*_t = t \), then the last period in which production occurs in an optimal \( t+1 \) period policy must be in the set \( t, t+1, \ldots t+1 \).

In the Example:

- We produce in period 4 for period 4 of a 4 period problem.
- We would never produce in period 3 for period 5 in a 5 period problem.

Wagner-Whitin Example (cont.)

**Step 5:** Only two choices, \( j^*_5 = 4, 5 \).

\[
Z^*_4 = \min \left\{ \begin{array}{ll}
Z^*_t + A_t + h_i D_s, & \text{produce in 4} \\
Z^*_t + A_t, & \text{produce in 5}
\end{array} \right. \\
= \min \left\{ \begin{array}{l}
170 + 100 + \text{I}(50) = 320 \\
270 + 100 = 370
\end{array} \right. \\
= 320
\]

\( j^*_5 = 4 \)

**Step 6:** Three choices, \( j^*_6 = 4, 5, 6 \).

And so on.
Wagner-Whitin Example Solution

<table>
<thead>
<tr>
<th>Last Period with Production</th>
<th>Planning Horizon ($t$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3</td>
</tr>
<tr>
<td>1</td>
<td>100 150 170</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
</tr>
<tr>
<td>4</td>
<td>270</td>
</tr>
<tr>
<td>5</td>
<td>360</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$Z_t$</td>
<td>100 150 170</td>
</tr>
<tr>
<td>$j_t$</td>
<td>1 1 1</td>
</tr>
</tbody>
</table>

Produce in period 1 for 1, 2, 3 (20 + 50 + 10 = 80 units)
Produce in period 4 for 4, 5, 6, 7 (50 + 50 + 10 + 20 = 130 units)
Produce in period 8 for 8, 9, 10 (40 + 20 + 10 = 90 units)

Wagner-Whitin Example Solution (cont.)

Optimal Policy:

• Produce in period 8 for 8, 9, 10 (40 + 20 + 30 = 90 units)
• Produce in period 4 for 4, 5, 6, 7 (50 + 50 + 10 + 20 = 130 units)
• Produce in period 1 for 1, 2, 3 (20 + 50 + 10 = 80 units)

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_t$</td>
<td>20</td>
<td>50</td>
<td>10</td>
<td>50</td>
<td>50</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>20</td>
<td>30</td>
<td>300</td>
</tr>
<tr>
<td>$Q_t$</td>
<td>80</td>
<td>0</td>
<td>0</td>
<td>130</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>$I_t$</td>
<td>60</td>
<td>10</td>
<td>0</td>
<td>80</td>
<td>30</td>
<td>20</td>
<td>0</td>
<td>50</td>
<td>30</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup cost</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>Holding cost</td>
<td>60</td>
<td>10</td>
<td>0</td>
<td>80</td>
<td>30</td>
<td>20</td>
<td>0</td>
<td>50</td>
<td>30</td>
<td>0</td>
<td>280</td>
</tr>
<tr>
<td>Total cost</td>
<td>160</td>
<td>10</td>
<td>0</td>
<td>180</td>
<td>30</td>
<td>20</td>
<td>0</td>
<td>150</td>
<td>30</td>
<td>0</td>
<td>580</td>
</tr>
</tbody>
</table>

Note: we produce in 7 for an 8 period problem, but this never comes into play in optimal solution.
Problems with Wagner-Whitin

1. Fixed setup costs.

2. Deterministic demand and production (no uncertainty)

3. Never produce when there is inventory (WW Property I).
   - safety stock (don't let inventory fall to zero)
   - random yields (can't produce for exact no. periods)

Statistical Reorder Point Models

When your pills get down to four,
Order more.

– Anonymous, from Hadley & Whitin
EOQ Assumptions

1. Instantaneous production. \( \rightarrow \) *EPL model relaxes this one*

2. Immediate delivery. \( \rightarrow \) *lags can be added to EOQ or other models*

3. **Deterministic demand** \( \rightarrow \) *newsvendor and \((Q,r)\) relax this one*

4. Constant demand. \( \rightarrow \) *WW model relaxes this one*

5. Known fixed setup costs. \( \rightarrow \) *can use constraint approach*

6. Single product or separable products. \( \rightarrow \) *Chapter 17 extends \((Q,r)\) to multiple product cases*

---

Modeling Philosophies for Handling Uncertainty

1. **Use deterministic model – adjust solution**
   - EOQ to compute order quantity, then add safety stock
   - deterministic scheduling algorithm, then add safety lead time

2. **Use stochastic model**
   - news vendor model
   - base stock and \((Q,r)\) models
   - variance constrained investment models
The Newsvendor Approach

Assumptions:
1. single period
2. random demand with known distribution
3. linear overage/shortage costs
4. minimum expected cost criterion

Examples:
- newspapers or other items with rapid obsolescence
- Christmas trees or other seasonal items
- capacity for short-life products

Newsvendor Model Notation

\[ X = \text{demand (in units), a random variable.} \]

\[ G(x) = P(X \leq x), \text{ cumulative distribution function of demand} \]
(assumed continuous)

\[ g(x) = \frac{d}{dx} G(x) = \text{density function of demand.} \]

\[ c_o = \text{cost (in dollars) per unit left over after demand is realized.} \]

\[ c_s = \text{cost (in dollars) per unit of shortage.} \]

\[ Q = \text{production/order quantity (in units); this is the decision variable.} \]
Newsvendor Model

Cost Function:

\[ Y(x) = \text{expected overage} + \text{expected shortage cost} \]

\[ = c_s E[\text{units over}]+ c_o E[\text{units short}] \]

\[ = c_s \int_0^\infty \max\{x,Q\} g(x) dx + c_o \int_0^\infty \max\{Q-x,0\} g(x) dx \]

\[ = c_s \int_0^Q (Q-x) g(x) dx + c_o \int_0^Q (x-Q) g(x) dx \]

Note: for any given day, we will be either over or short, not both. But in expectation, overage and shortage can both be positive.

Frank Matejcik  SD School of Mines & Technology

Newsvendor Model (cont.)

Optimal Solution: taking derivative of \( Y(Q) \) with respect to \( Q \), setting equal to zero, and solving yields:

\[ G(Q^*) = P\{X \leq Q^*\} = \frac{c_s}{c_o + c_s} \]

Critical Ratio is probability stock covers demand

Notes:

\( Q^* \downarrow c_o \)

\( Q^* \uparrow c_s \)

Frank Matejcik  SD School of Mines & Technology
Newsvendor Example – T Shirts

Scenario:
- Demand for T-shirts is exponential with mean 1000 (i.e., \( G(x) = P(X \leq x) = 1 - e^{-x/1000} \)). (Note - this is an odd demand distribution; Poisson or Normal would probably be better modeling choices.)
- Cost of shirts is $10.
- Selling price is $15.
- Unsold shirts can be sold off at $8.

Model Parameters:
- \( c_s = 15 - 10 = $5 \)
- \( c_o = 10 - 8 = $2 \)

Newsvendor Example – T Shirts (cont.)

Solution:
- \( G(Q^*) = 1 - e^{-\frac{Q^*}{1000}} = \frac{c_s}{c_o + c_s} = \frac{5}{2 + 5} = 0.714 \)
- \( Q^* = 1.253 \)

Sensitivity: If \( c_o = $10 \) (i.e., shirts must be discarded) then
- \( G(Q^*) = 1 - e^{-\frac{Q^*}{1000}} = \frac{c_s}{c_o + c_s} = \frac{5}{10 + 5} = 0.333 \)
- \( Q^* = 405 \)
Newsvendor Model with Normal Demand

Suppose demand is normally distributed with mean $\mu$ and standard deviation $\sigma$. Then the critical ratio formula reduces to:

$$ G(Q^*) = \Phi\left( \frac{Q^* - \mu}{\sigma} \right) = \frac{c_i}{c_o + c_i} $$

$$ \frac{Q^* - \mu}{\sigma} = z \quad \text{where} \quad \Phi(z) = \frac{c_i}{c_o + c_i} $$

$$ Q^* = \mu + z\sigma \quad \text{Note:} \; Q^* \; \text{increases in both} \; \mu \; \text{and} \; \sigma \; \text{if} \; z \; \text{is positive} \; (i.e., \; \text{if ratio is greater than} \; 0.5). $$

Multiple Period Problems

**Difficulty:** Technically, Newsvendor model is for a single period.

**Extensions:** But Newsvendor model can be applied to multiple period situations, provided:

- demand during each period is iid, distributed according to $G(x)$
- there is no setup cost associated with placing an order
- stockouts are either lost or backordered

**Key:** make sure $c_o$ and $c_i$ appropriately represent overage and shortage cost.
Example

Scenario:
• GAP orders a particular clothing item every Friday
• mean weekly demand is 100, std dev is 25
• wholesale cost is $10, retail is $25
• holding cost has been set at $0.5 per week (to reflect obsolescence, damage, etc.)

Problem: how should they set order amounts?

Example (cont.)

Newsvendor Parameters:
\[ c_0 = \$0.5 \]
\[ c_s = \$15 \]

Solution:
\[ G(Q^*) = \frac{15}{0.5 + 15} = 0.9677 \]
\[ \Phi \left( \frac{Q - 100}{25} \right) = 0.9677 \]
\[ \frac{Q - 100}{25} = 1.85 \]
\[ Q = 100 + 1.85(25) = 146 \]

Every Friday, they should order-up-to 146, that is, if there are \( x \) on hand, then order \( 146-x \).
Newsvendor Takeaways

- Inventory is a hedge against demand uncertainty.
- Amount of protection depends on “overage” and “shortage” costs, as well as distribution of demand.
- If shortage cost exceeds overage cost, optimal order quantity generally increases in both the mean and standard deviation of demand.

The \((Q,r)\) Approach

Assumptions:
1. Continuous review of inventory.
2. Demands occur one at a time.
3. Unfilled demand is backordered.
4. Replenishment lead times are fixed and known.

Decision Variables:
- **Reorder Point**: \(r\) – affects likelihood of stockout (safety stock).
- **Order Quantity**: \(Q\) – affects order frequency (cycle inventory).
Inventory vs Time in \((Q,r)\) Model

Base Stock Model Assumptions

1. There is no fixed cost associated with placing an order.

2. There is no constraint on the number of orders that can be placed per year.

That is, we can replenish one at a time \((Q=1)\).
Base Stock Notation

- $Q = 1$, order quantity (fixed at one)
- $r = \text{reorder point}$
- $R = r + 1$, base stock level
- $l = \text{delivery lead time}$
- $\theta = \text{mean demand during } l$
- $\sigma = \text{std dev of demand during } l$
- $p(x) = \text{Prob}\{\text{demand during lead time } \leq x\}$
- $G(x) = \text{Prob}\{\text{demand during lead time } < x\}$
- $h = \text{unit holding cost}$
- $b = \text{unit backorder cost}$
- $S(R) = \text{average fill rate (service level)}$
- $B(R) = \text{average backorder level}$
- $I(R) = \text{average on-hand inventory level}$

Inventory Balance Equations

Balance Equation:

inventory position = on-hand inventory - backorders + orders

Under Base Stock Policy

inventory position = $R$
Inventory Profile for Base Stock System (R=5)

Service Level (Fill Rate)

Let:

\[ X = \text{(random) demand during lead time} \]

so \( E[X] = 0 \). Consider a specific replenishment order. Since inventory position is always \( R \), the only way this item can stock out is if \( X \geq R \).

Expected Service Level:

\[ S(R) = P(X < R) = \begin{cases} G(R), & \text{if } G \text{ is continuous} \\ G(R-1) = G(r), & \text{if } G \text{ is discrete} \end{cases} \]
Backorder Level

Note: At any point in time, number of orders equals number demands during last \( l \) time units \( (X) \) so from our previous balance equation:

\[
R = \text{on-hand inventory} - \text{backorders} + \text{orders} \\
\text{on-hand inventory} - \text{backorders} = R - X
\]

Note: on-hand inventory and backorders are never positive at the same time, so if \( X = x \), then

\[
\text{backorders} = \begin{cases} 
0, & \text{if } x < R \\
x - R, & \text{if } x \geq R \\
\end{cases}
\]

Expected Backorder Level:

\[
B(R) = \sum_{x=R}^{\infty} (x - R)p(x) = \theta p(R) + (\theta - R)[1 - G(R)]
\]

Inventory Level

Observe:

- on-hand inventory - backorders = \( R \cdot X \)
- \( E[X] = \theta \) \textit{ from data}
- \( E[\text{backorders}] = B(R) \) \textit{ from previous slide}

Result:

\[
I(R) = R - \theta + B(R)
\]
Base Stock Example

\[ I = \text{one month} \]

\[ \theta = 10 \text{ units (per month)} \]

Assume Poisson demand, so

\[ G(x) = \sum_{k=0}^{\infty} p(k) = \sum_{k=0}^{\infty} \left( \frac{\theta^k e^{-\theta}}{k!} \right) \]

Note: Poisson demand is a good choice when no variability data is available.

---

Base Stock Example Calculations

<table>
<thead>
<tr>
<th>R</th>
<th>p(R)</th>
<th>G(R)</th>
<th>B(R)</th>
<th>R</th>
<th>p(R)</th>
<th>G(R)</th>
<th>B(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>10.000</td>
<td>12</td>
<td>0.095</td>
<td>0.792</td>
<td>0.531</td>
</tr>
<tr>
<td>1</td>
<td>0.000</td>
<td>0.000</td>
<td>9.000</td>
<td>13</td>
<td>0.073</td>
<td>0.864</td>
<td>0.322</td>
</tr>
<tr>
<td>2</td>
<td>0.002</td>
<td>0.003</td>
<td>8.001</td>
<td>14</td>
<td>0.052</td>
<td>0.917</td>
<td>0.187</td>
</tr>
<tr>
<td>3</td>
<td>0.008</td>
<td>0.010</td>
<td>7.003</td>
<td>15</td>
<td>0.035</td>
<td>0.951</td>
<td>0.103</td>
</tr>
<tr>
<td>4</td>
<td>0.019</td>
<td>0.029</td>
<td>6.014</td>
<td>16</td>
<td>0.022</td>
<td>0.973</td>
<td>0.055</td>
</tr>
<tr>
<td>5</td>
<td>0.038</td>
<td>0.067</td>
<td>5.043</td>
<td>17</td>
<td>0.013</td>
<td>0.986</td>
<td>0.028</td>
</tr>
<tr>
<td>6</td>
<td>0.063</td>
<td>0.130</td>
<td>4.110</td>
<td>18</td>
<td>0.007</td>
<td>0.993</td>
<td>0.013</td>
</tr>
<tr>
<td>7</td>
<td>0.090</td>
<td>0.220</td>
<td>3.240</td>
<td>19</td>
<td>0.004</td>
<td>0.997</td>
<td>0.006</td>
</tr>
<tr>
<td>8</td>
<td>0.113</td>
<td>0.333</td>
<td>2.460</td>
<td>20</td>
<td>0.002</td>
<td>0.998</td>
<td>0.003</td>
</tr>
<tr>
<td>9</td>
<td>0.125</td>
<td>0.458</td>
<td>1.793</td>
<td>21</td>
<td>0.001</td>
<td>0.999</td>
<td>0.001</td>
</tr>
<tr>
<td>10</td>
<td>0.125</td>
<td>0.583</td>
<td>1.251</td>
<td>22</td>
<td>0.000</td>
<td>0.999</td>
<td>0.000</td>
</tr>
<tr>
<td>11</td>
<td>0.114</td>
<td>0.697</td>
<td>0.834</td>
<td>23</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Base Stock Example Results

Service Level: For fill rate of 90%, we must set $R-1= r = 14$, so $R=15$ and safety stock $s = r-\theta = 4$. Resulting service is 91.7%.

Backorder Level:

$$B(R) = B(15) = 0.103$$

Inventory Level:

$$I(R) = R - \theta + B(R) = 15 - 10 + 0.103 = 5.103$$

“Optimal” Base Stock Levels

Objective Function:

$$Y(R) = hI(R) + bB(R)$$

$$= h(R-\theta + B(R)) + bB(R)$$

$$= h(R- \theta) + (h+b)B(R)$$

Solution: if we assume $G$ is continuous, we can use calculus to get

$$G(R^*) = \frac{b}{h+b}$$

Implication: set base stock level so fill rate is $b/(h+b)$.

Note: $R^*$ increases in $b$ and decreases in $h$. 
Base Stock Normal Approximation

If $G$ is normal($\theta$, $\sigma$), then

$$G(R^*) = \frac{b}{h+b} \iff \Phi\left(\frac{R^*-\theta}{\sigma}\right) = z$$

where $\Phi(z) = b/(h+b)$. So

$$R^* = \theta + z \sigma$$

*Note: $R^*$ increases in $\theta$ and also increases in $\sigma$ provided $z>0$."

“Optimal” Base Stock Example

**Data:** Approximate Poisson with mean 10 by normal with mean 10 units/month and standard deviation $\sqrt{10} = 3.16$ units/month. Set $h=$$15$, $b=$$25$.

**Calculations:**

$$\frac{b}{h+b} = \frac{25}{15+25} = 0.625$$

since $\Phi(0.32) = 0.625$, $z=0.32$ and hence

$$R^* = \theta + z\sigma = 10 + 0.32(3.16) = 11.01 \approx 11$$

**Observation:** from previous table fill rate is $G(10) = 0.583$, so maybe backorder cost is too low.
Inventory Pooling

**Situation:**
- $n$ different parts with lead time demand normal($\theta$, $\sigma$)
- $z=2$ for all parts (i.e., fill rate is around 97.5%)

**Specialized Inventory:**
- base stock level for each item = $\theta + 2\sigma$
- total safety stock = $2n\sigma$

**Pooled Inventory:** suppose parts are substitutes for one another
- lead time demand is normal ($n\theta, \sqrt{n}\sigma$)
- base stock level (for same service) = $n\theta + 2\sqrt{n}\sigma$
- ratio of safety stock to specialized safety stock = $1/\sqrt{n}$

Effect of Pooling on Safety Stock

**Conclusion:** cycle stock is not affected by pooling, but safety stock falls dramatically. So, for systems with high safety stock, pooling (through product design, late customization, etc.) can be an attractive strategy.
Pooling Example

- PC’s consist of 6 components (CPU, HD, CD ROM, RAM, removable storage device, keyboard)
- 3 choices of each component: $3^6 = 729$ different PC’s
- Each component costs $150 ($900 material cost per PC)
- Demand for all models is Poisson distributed with mean 100 per year
- Replenishment lead time is 3 months (0.25 years)
- Use base stock policy with fill rate of 99%

Pooling Example - Stock PC’s

**Base Stock Level for Each PC:** $\theta = 100 \times 0.25 = 25$, so using Poisson formulas,

$$G(R-1) \geq 0.99 \quad \Rightarrow \quad R = 38 \text{ units}$$

**On-Hand Inventory for Each PC:**

$$I(R) = R - \theta + B(R) = 38 - 25 + 0.0138 = 13.0138 \text{ units}$$

**Total On-Hand Inventory:**

$$13.0138 \times 729 \times 900 = 8,538,358$$
Pooling Example - Stock Components

Necessary Service for Each Component:
\[ S = (0.99)^{1/6} = 0.9983 \]

Base Stock Level for Components:
\[ \theta = (100 \times 729/3)0.25 = 6075, \text{ so} \]
\[ G(R-I) \geq 0.9983 \quad \implies \quad R = 6306 \]

On-Hand Inventory Level for Each Component:
\[ I(R) = R - \theta + B(R) = 6306-6075+0.0363 = 231.0363 \text{ units} \]

Total On-Hand Inventory:
\[ 231.0363 \times 18 \times 150 = 623,798 \]

93% reduction!

Base Stock Insights

1. Reorder points control prob of stockouts by establishing safety stock.

2. To achieve a given fill rate, the required base stock level (and hence safety stock) is an increasing function of mean and (provided backorder cost exceeds shortage cost) std dev of demand during replenishment lead time.

3. The “optimal” fill rate is an increasing in the backorder cost and a decreasing in the holding cost. We can use either a service constraint or a backorder cost to determine the appropriate base stock level.

4. Base stock levels in multi-stage production systems are very similar to kanban systems and therefore the above insights apply.

5. Base stock model allows us to quantify benefits of inventory pooling.
The Single Product \((Q,r)\) Model

**Motivation:** Either
1. Fixed cost associated with replenishment orders and cost per backorder.
2. Constraint on number of replenishment orders per year and service constraint.

**Objective:** Under (1)

\[
\min_{Q,r} \left\{ \text{fixed setup cost + holding cost + backorder cost} \right\}
\]

As in EOQ, this makes batch production attractive.

---

Summary of \((Q,r)\) Model Assumptions

1. One-at-a-time demands.
2. Demand is uncertain, but stationary over time and distribution is known.
3. Continuous review of inventory level.
4. Fixed replenishment lead time.
5. Constant replenishment batch sizes.
6. Stockouts are backordered.
(Q,r) Notation

\( D \) = expected demand per year
\( \ell \) = replenishment lead time (assumed constant)
\( X \) = (random) demand during replenishment lead time
\( \theta = E[X] \) = expected demand during replenishment lead time
\( \sigma \) = standard deviation of demand during replenishment lead time
\( p(x) = P(X = x) \) = pmf of demand during lead time
\( G(x) = P(X \leq x) \) = cdf of demand during lead time
\( A \) = fixed cost per order
\( c \) = unit cost of an item
\( h \) = annual unit holding cost
\( k \) = cost per stockout
\( b \) = annual unit backorder cost

(\(Q, r\) Notation (cont.))

Decision Variables:
\( Q \) = order quantity
\( r \) = reorder point
\( s = r - \theta = \text{safety stock implied by } r \)

Performance Measures:
\( F(Q) \) = average order frequency
\( S(Q, r) \) = average service level (fill rate)
\( B(Q, r) \) = average backorder level
\( I(Q, r) \) = average inventory level
Inventory and Inventory Position for $Q=4$, $r=4$

<table>
<thead>
<tr>
<th>Time</th>
<th>Inventory Position</th>
<th>Net Inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Inventory Position uniformly distributed between $r+1=5$ and $r+Q=8$

Costs in $(Q,r)$ Model

- **Fixed Setup Cost**: $AF(Q)$
- **Stockout Cost**: $kD(1-S(Q,r))$, where $k$ is cost per stockout
- **Backorder Cost**: $bB(Q,r)$
- **Inventory Carrying Costs**: $cI(Q,r)$
Fixed Setup Cost in \((Q,r)\) Model

**Observation:** since the number of orders per year is \(D/Q\).

\[
F(Q) = \frac{D}{Q}
\]

Stockout Cost in \((Q,r)\) Model

**Key Observation:** inventory position is uniformly distributed between \(r+1\) and \(r+Q\). So, service in \((Q,r)\) model is weighted sum of service in base stock model.

**Result:**

\[
S(Q, r) = \frac{1}{Q} \sum_{x=r}^{r+Q} G(x-1) = \frac{1}{Q} [G(r) + \cdots + G(r+Q-1)]
\]

\[
S(Q, r) = 1 - \frac{1}{Q} [B(r) - B(r+Q)]
\]

*Note: this form is easier to use in spreadsheets because it does not involve a sum.*
Service Level Approximations

Type I (base stock):

\[ S(Q, r) \approx G(r) \]

*Note: computes number of stockouts per cycle, underestimates \( S(Q, r) \)*

Type II:

\[ S(Q, r) = 1 - \frac{B(r)}{Q} \]

*Note: neglects \( B(r, Q) \) term, underestimates \( S(Q, r) \)*

---

Backorder Costs in \((Q, r)\) Model

**Key Observation:** \( B(Q, r) \) can also be computed by averaging base stock backorder level function over the range \([r+1, r+Q]\).

**Result:**

\[ B(Q, r) = \frac{1}{Q} \sum_{x=r+1}^{r+Q} B(x) = \frac{1}{Q} \left[ B(r+1) + \cdots + B(r+Q) \right] \]

**Notes:**
1. \( B(Q, r) \approx B(r) \) is a base stock approximation for backorder level.
2. If we can compute \( B(x) \) (base stock backorder level function), then we can compute stockout and backorder costs in \((Q, r)\) model.
Inventory Costs in $(Q,r)$ Model

**Approximate Analysis:** on average inventory declines from $Q+s$ to $s+1$ so

\[
I(Q,r) \approx \frac{(Q+s) + (s+1)}{2} = \frac{Q+1}{2} + s = \frac{Q+1}{2} + r - \theta
\]

**Exact Analysis:** this neglects backorders, which add to average inventory since on-hand inventory can never go below zero. The corrected version turns out to be

\[
I(Q,r) = \frac{Q+1}{2} + r - \theta + B(Q,r)
\]

Inventory vs Time in $(Q,r)$ Model

- Expected Inventory
- Actual Inventory
- Exact $I(Q,r) = \text{Approx } I(Q,r) + B(Q,r)$
- Approx $I(Q,r)$

Time
Expected Inventory Level for $Q=4, r=4, \theta=2$

(\(Q,r\)) Model with Backorder Cost

Objective Function:

\[ Y(Q,r) = \frac{D}{Q} \cdot A + bB(Q,r) + hI(Q,r) \]

Approximation: \(B(Q,r)\) makes optimization complicated because it depends on both \(Q\) and \(r\). To simplify, approximate with base stock backorder formula, \(B(r)\):

\[ Y(Q,r) \approx \tilde{Y}(Q,r) = \frac{D}{Q} \cdot A + bB(r) + h\left(\frac{Q+1}{2}\right) + r - \theta + B(r) \]
Results of Approximate Optimization

Assumptions:

• $Q,r$ can be treated as continuous variables
• $G(x)$ is a continuous cdf

Results:

$Q^* = \sqrt{\frac{2AD}{h}}$ \hspace{1cm} \text{Note: this is just the EOQ formula}

$G(r^*) = \frac{b}{h+b} \Rightarrow r^* = \theta + z\sigma$ \hspace{1cm} \text{Note: this is just the base stock formula}

if $G$ is normal($\theta,\sigma$), where \( \Phi(z) = b/(h+b) \)

(Q,r) Example

Stocking Repair Parts:

$D = 14$ units per year
$C = $150 per unit
$h = 0.1 \times 150 + 10 = $25 per unit
$l = 45$ days
$\theta = (14 \times 45)/365 = 1.726$ units during replenishment lead time
$A = $10
$b = $40

Demand during lead time is Poisson
### Values for Poisson(θ) Distribution

<table>
<thead>
<tr>
<th>r</th>
<th>p(r)</th>
<th>G(r)</th>
<th>B(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.178</td>
<td>0.178</td>
<td>1.726</td>
</tr>
<tr>
<td>1</td>
<td>0.307</td>
<td>0.485</td>
<td>0.904</td>
</tr>
<tr>
<td>2</td>
<td>0.265</td>
<td>0.750</td>
<td>0.389</td>
</tr>
<tr>
<td>3</td>
<td>0.153</td>
<td>0.903</td>
<td>0.140</td>
</tr>
<tr>
<td>4</td>
<td>0.066</td>
<td>0.969</td>
<td>0.042</td>
</tr>
<tr>
<td>5</td>
<td>0.023</td>
<td>0.991</td>
<td>0.011</td>
</tr>
<tr>
<td>6</td>
<td>0.007</td>
<td>0.998</td>
<td>0.003</td>
</tr>
<tr>
<td>7</td>
<td>0.002</td>
<td>1.000</td>
<td>0.001</td>
</tr>
<tr>
<td>8</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>9</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>10</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

### Calculations for Example

\[ Q^* = \sqrt{\frac{2AD}{h}} = \sqrt{\frac{2(10)(14)}{15}} = 4.3 \approx 4 \]

\[ \frac{b}{h+b} = \frac{40}{25+40} = 0.615 \]

\[ \Phi(0.29) = 0.615, \text{ so } z = 0.29 \]

\[ r^* = \theta + z\sigma = 1.726 + 0.29(1.314) = 2.107 \approx 2 \]
Performance Measures for Example

\[ F(Q^*) = \frac{D}{Q^*} = \frac{14}{4} = 3.5 \]

\[ S(Q^*, r^*) = 1 - \frac{1}{Q^*}[B(r^*) - B(r^* + Q^*)] = 1 - \frac{1}{Q^*}[B(2) - B(2 + 4)] \]
\[ = 1 - \frac{1}{4}[0.389 - 0.003] = 0.904 \]

\[ B(Q^*, r^*) = \frac{1}{Q^*} \sum_{x=r^*}^{Q^*} B(x) = \frac{1}{Q^*}[B(3) + B(4) + B(5) + B(6)] \]
\[ = \frac{1}{4}[0.140 + 0.042 + 0.011 + 0.003] = 0.049 \]

\[ I(Q^*, r^*) = \frac{Q^* + 1}{2} + r^* - \theta + B(Q^*, r^*) = \frac{4 + 1}{2} + 2 - 1.726 + 0.049 = 2.823 \]

Observations on Example

- Orders placed at rate of 3.5 per year
- Fill rate fairly high (90.4%)
- Very few outstanding backorders (0.049 on average)
- Average on-hand inventory just below 3 (2.823)
Varying the Example

Change: suppose we order twice as often so $F=7$ per year, then $Q=2$ and:

$$S(Q, r) = 1 - \frac{1}{Q} [B(r) - B(r + Q)] = 1 - \frac{1}{2} [0.389 - 0.042] = 0.826$$

which may be too low, so increase $r$ from 2 to 3:

$$S(Q, r) = 1 - \frac{1}{Q} [B(r) - B(r + Q)] = 1 - \frac{1}{2} [0.140 - 0.011] = 0.936$$

This is better. For this policy $(Q=2, r=4)$ we can compute $B(2,3)=0.026, I(Q,r)=2.80$.

Conclusion: this has higher service and lower inventory than the original policy $(Q=4, r=2)$. But the cost of achieving this is an extra 3.5 replenishment orders per year.

(Q,r) Model with Stockout Cost

Objective Function:

$$Y(Q, r) = \frac{D}{Q} A + kD(1 - S(Q, r)) + hI(Q, r)$$

Approximation: Assume we still use EOQ to compute $Q^*$ but replace $S(Q, r)$ by Type II approximation and $B(Q, r)$ by base stock approximation:

$$Y(Q, r) \approx \tilde{Y}(Q, r) = \frac{D}{Q} A + kD \frac{B(r)}{Q} + h \left( \frac{Q+1}{2} + r - \theta + B(r) \right)$$
Results of Approximate Optimization

Assumptions:
- $Q, r$ can be treated as continuous variables
- $G(x)$ is a continuous cdf

Results:
\[
Q^* = \sqrt{\frac{2AD}{h}} \quad \text{Note: this is just the EOQ formula}
\]
\[
G(r^*) = \frac{kD}{kD + hQ} \quad \Rightarrow \quad r^* = \theta + z\sigma \quad \text{Note: another version of base stock formula (only } z \text{ is different)}
\]
if $G$ is normal($\theta, \sigma$), where $\Phi(z) = kD/(kD+hQ)$

Backorder vs. Stockout Model

Backorder Model
- when real concern is about stockout time
- because $B(Q, r)$ is proportional to time orders wait for backorders
- useful in multi-level systems

Stockout Model
- when concern is about fill rate
- better approximation of lost sales situations (e.g., retail)

Note:
- We can use either model to generate frontier of solutions
- Keep track of all performance measures regardless of model
- B-model will work best for backorders, S-model for stockouts
Lead Time Variability

**Problem:** replenishment lead times may be variable, which increases variability of lead time demand.

**Notation:**
- $L$ = replenishment lead time (days), a random variable
- $\mu$ = $E[L]$ = expected replenishment lead time (days)
- $\sigma_L$ = std dev of replenishment lead time (days)
- $D_t$ = demand on day $t$, a random variable, assumed independent and identically distributed
- $d$ = $E[D_t]$ = expected daily demand
- $\sigma_D$ = std dev of daily demand (units)

Including Lead Time Variability in Formulas

**Standard Deviation of Lead Time Demand:**

*if demand is Poisson*

$$
\sigma = \sqrt{\epsilon \sigma_D^2 + d^2 \sigma_L^2} = \sqrt{\theta + d^2 \sigma^2_L}
$$

Inflation term due to lead time variability

**Modified Base Stock Formula (Poisson demand case):**

$$
R = \theta + z\sigma = \theta + z\sqrt{\theta + d^2 \sigma^2_L}
$$

Note: $\sigma$ can be used in any base stock or $(Q,r)$ formula as before. In general, it will inflate safety stock.
Single Product \((Q, r)\) Insights

Basic Insights:
- Safety stock provides a buffer against stockouts.
- Cycle stock is an alternative to setups/orders.

Other Insights:
1. Increasing \(D\) tends to increase optimal order quantity \(Q\).
2. Increasing \(\theta\) tends to increase the optimal reorder point. (Note: either increasing \(D\) or \(l\) increases \(\theta\).)
3. Increasing the variability of the demand process tends to increase the optimal reorder point (provided \(z > 0\)).
4. Increasing the holding cost tends to decrease the optimal order quantity and reorder point.