Problem 1.7.7 Solution

Let $T = \{ x \mid -x \in S \}$. $S$ is nonempty, so $T$ is nonempty. By hypothesis $S$ has a lower bound $L_0$, so $L_0 \leq x$ for all $x \in S$. But then for all $t \in T$, we know that $-t \in S$, therefore $L_0 \leq -t$ and so $-L_0 \geq t$. This makes $-L_0$ an upper bound of $T$, so the Completeness Axiom guarantees the existence of $T_0$, the least upper bound of $T$.

Claim: $-T_0$ is the greatest lower bound of $S$. We must show (1) that $-T_0$ is a lower bound of $S$ and (2) that of all lower bounds of $S$, $-T_0$ is greatest.

Let $x$ be any element of $S$. Then $-x \in T$, so $-x \leq T_0$ and $x \geq -T_0$. Therefore $-T_0$ is a lower bound for $S$.

Next, choose any lower bound $L$ of $S$. For any $t \in T$, we have $-t \in S$, so $L \leq -t$, which means $-L \geq t$. This makes $-L$ an upper bound of $T$, so $T_0 \leq -L$ (the least upper bound is smaller than any other upper bound). But then $-T_0 \geq L$. Since $-T_0$ is a lower bound for $S$ and is greater than or equal to any other lower bound for $S$, it follows that $-T_0$ is the greatest lower bound for $S$. 