I have most of your exams graded. Expect to have the study guide post and solutions sent out tonight.

Agenda

Factory Physics

Chapter 16: Aggregate and Workforce Planning
Chapter 17: Supply Chain Management
(New Assignment Chapter 16 problems 1-4
Chapter 17 problem 1)
### Tentative Schedule

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<td>13, 14 C13: C14:1,2</td>
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### Aggregate Planning

*And I remember misinformation followed us like a plague,*

*Nobody knew from time to time if the plans were changed.*

– Paul Simon

---

### Aggregate Planning Issues

**Role of Aggregate Planning**
- Long-term planning function
- Strategic preparation for tactical actions

**Aggregate Planning Issues**
- **Production Smoothing**: inventory build-ahead
- **Product Mix Planning**: best use of resources
- **Staffing**: hiring, firing, training
- **Procurement**: supplier contracts for materials, components
- **Sub-Contracting**: capacity vending
- **Marketing**: promotional activities
Hierarchical Production Planning

Basic Aggregate Planning

Problem: project production of single product over planning horizon.

Motivation for Study:
- mechanics and value of LP as a tool
- intuition of production smoothing

Inputs:
- demand forecast (over planning horizon)
- capacity constraints
- unit profit
- inventory carrying cost rate

A Simple AP Model

Notation:
- $t$ = an index of the time periods, $t = 1, \ldots, T$
- $d_t$ = demand in period $t$
- $c_t$ = capacity in period $t$
- $r$ = unit profit (not including holding cost)
- $h$ = cost to hold one unit of inventory for one period.
- $X_t$ = quantity produced during period $t$
- $S_t$ = quantity sold during period $t$
- $I_t$ = inventory at the end of period $t$
A Simple AP Model (cont.)

Formulation

\[ \max \sum_{t=1}^{T} (S_t - h_d t) \]

subject to

\[ S_t \leq d_t \quad t = 1, \ldots, T \] demand
\[ X_t \leq c_t \quad t = 1, \ldots, T \] capacity
\[ I_t = I_{t-1} + X_t - S_t \quad t = 1, \ldots, T \] inventory balance
\[ X_t, S_t, I_t \geq 0 \quad t = 1, \ldots, T \] non-negativity

A Simple AP Example

Data:
\[ r = $10 \]
\[ h = $1 \]
\[ i_t = 0 \]
\[ c_t = -100, t = 1, \ldots, 5 \]
\[ d_t = 80, 100, 120, 140, 90, 140 \]

Optimal Solution:

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>20</td>
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<td>0</td>
<td>20</td>
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</tr>
</tbody>
</table>

A Simple AP Example (cont.)

Interpretation

- solution
- shadow prices
- allowable increases / decreases
Product Mix Planning

Problem: determine most profitable mix over planning horizon

Motivation for Study:
- linking marketing/promotion to logistics
- Bottleneck identification

Inputs:
- demand forecast by product (family?), may be ranges
- unit hour data
- capacity constraints
- unit profit by product
- holding cost

Basic Verbal Formulation

\[
\text{maximize } \text{profit} \\
\text{subject to:} \\
\text{production } \leq \text{ capacity, } \text{at all workstations} \\
\text{sales } \leq \text{ demand, } \text{for all products} \\
\text{Note: we will need some technical constraints} \\
\text{to ensure that variables represent reality.}
\]

Product Mix Notation

\begin{align*}
i & \text{ an index of product, } i = 1, \ldots, m \\
j & \text{ an index of workstation, } j = 1, \ldots, n \\
t & \text{ an index of period, } t = 1, \ldots, T \\
d_i^t & \text{ maximum demand for product } i \text{ in period } t \\
d_i^m & \text{ minimum sales allowed for product } i \text{ in period } t \\
w_j & \text{ time required on workstation } j \text{ to produce one unit of product } i \\
h_i & \text{ capacity of workstation } j \text{ in period } t \\
\pi_i & \text{ net profit from one unit of product } i \\
b_i & \text{ cost to hold one unit of } i \text{ for one period } t \\
X_i^t & \text{ amount of product } i \text{ produced in period } t \\
S_i^t & \text{ amount of product } i \text{ sold in period } t \\
I_i & \text{ inventory of product } i \text{ at end of } t
\end{align*}
### Product Mix Formulation

\[
\max \sum_{i=1}^{n} \sum_{t} r_i S_{it} - h_i I_{it} \quad \text{sales revenue - holding cost}
\]

subject to

\[
\begin{align*}
\sum_{i=1}^{n} a_{it} X_{it} & \leq c_{jt} \quad \text{for all } i, t \quad \text{demand} \\
\sum_{t} a_{it} X_{it} & \leq c_{jt} \quad \text{for all } j, t \quad \text{capacity} \\
I_{it} &= I_{i,t-1} + X_{it} - S_{it} \quad \text{for all } i, t \quad \text{inventory balance} \\
X_{it}, S_{it}, I_{it} & \geq 0 \quad \text{for all } i, t \quad \text{non-negativity}
\end{align*}
\]

### Product Mix (Goldratt) Example

**Assumptions:**
- Two products, \( P \) and \( Q \)
- Constant weekly demand, cost, capacity, etc.
- **Objective:** Maximize weekly profit

**Data:**

<table>
<thead>
<tr>
<th>Product</th>
<th>( P )</th>
<th>( Q )</th>
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</thead>
<tbody>
<tr>
<td>Selling price</td>
<td>$90</td>
<td>$100</td>
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<tr>
<td>Raw Material Cost</td>
<td>$30</td>
<td>$40</td>
</tr>
<tr>
<td>Max Weekly Sales</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter A</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter B</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter C</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter D</td>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

### A Cost Approach

**Unit Profit**
- Product \( P \): $45
- Product \( Q \): $40

**Maximum Production of \( Q \): 50 units**

**Available Capacity for Producing \( P \):**
- 2400 - 15 (150) = 1,950 minutes on Workcenter A
- 2400 - 30 (150) = 2,100 minutes on Workcenter B
- 2400 - 5 (150) = 2,150 minutes on Workcenter C
- 2400 - 5 (150) = 2,150 minutes on Workcenter D

**Maximum Production of \( P \): 900/15 = 60 units**

**Net Weekly Profit:** $45 \times 60 + $60 \times 50 - $5,000 = $700
A Bottleneck Approach

Identifying the Bottleneck: Workcenter B, because
15 (100) + 35 (50) = 2,000 minutes on workcenter A
15 (100) + 35 (50) = 1,750 minutes on workcenter B
15 (100) + 5 (50) = 1,750 minutes on workcenter C
15 (100) + 5 (50) = 1,750 minutes on workcenter D

Profit per Minute of Bottleneck Time used:
$45/15 = $3 per minute spent processing P
$60/35 = $1.71 per minute spent processing Q

Maximum Production of P: 100 units
Maximum Production of Q: 900/30 = 30 units

Net Weekly Profit: $45 \times 100 + $60 \times 30 - $5,000 = $1,300

A Modified Example

Changes: in processing times on workcenters B and D.

Data:

<table>
<thead>
<tr>
<th>Product</th>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling price</td>
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<td>$100</td>
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<tr>
<td>Raw Material Cost</td>
<td>$45</td>
<td>$40</td>
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<tr>
<td>Max Weekly Sales</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter A</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter B</td>
<td>15</td>
<td>35</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter C</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>Minutes per unit on Workcenter D</td>
<td>25</td>
<td>14</td>
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</table>

A Bottleneck Approach

Identifying the Bottleneck: Workcenter B, because
15 (100) + 35 (50) = 2,000 minutes on workcenter A
15 (100) + 35 (50) = 1,750 minutes on workcenter B
15 (100) + 5 (50) = 1,750 minutes on workcenter C
15 (100) + 5 (50) = 1,750 minutes on workcenter D

Bottleneck at B:
$45/15 = $3 per minute spent processing P
$60/35 = $1.71 per minute spent processing Q

Maximum Production of P: 2400/25 = 96 units
Maximum Production of Q: 0 units

Net Weekly Profit: $45 \times 96 - $5,000 = $680
Bottleneck Approach (cont.)

Bottleneck at D:
$45/25 = 1.80 \text{ per minute spent processing } P$
$60/14 = 4.29 \text{ per minute spent processing } Q$

Maximum Production of $Q$: $2400/35 = 68.57 > 50$, produce 50

Available time on Bottleneck:
$2400 - 14(50) = 1,700 \text{ minutes on workcenter D}$

Maximum Production of $P$: $1700/25 = 68$ units

Net Weekly Profit:
$45 \times 43 + 60 \times 50 - 5000 = -65$

An LP Approach

Formulation:
max $45X_P + 60X_Q - 5000$
subject to:
$15X_P + 10X_Q \leq 2400$
$15X_P + 55X_Q \leq 2400$
$15X_P + 5X_Q \leq 2400$
$25X_P + 14X_Q \leq 2400$

Solution:
Optimal Objective = $557.94$
$X_P = 36.09$
$X_Q = 75.79$

Net Weekly Profit: Round solution down (still feasible) to:
$X_P = 36$
$X_Q = 75$

To get $45 \times 75 + 60 \times 36 - 5,000 = 535$

Extensions to Basic Product Mix Model

Other Resource Constraints:

Notation:

$b_{jt} = \text{units of resource } j \text{ required per unit of product } i$
$t_{jt} = \text{number of units of resource } j \text{ available in period } i$
$X_{it} = \text{amount of product } i \text{ produced in period } t$

Constraint for Resource $j$: $\sum_{i=1}^{m} b_{jt} X_{it} \leq t_{jt}$

Utilization Matching: Let $q$ represent fraction of rated capacity we are willing to run on resource $j$.

$\sum_{i=1}^{m} a_{jt} X_{it} \leq q \times b_{jt}$ for all $j, t$
Extensions to Basic Product Mix Model (cont.)

Backorders:
• Substitute $L^+_t = L^+_t - L^-_t$
• Allow $L^-_t$ to become positive or negative
• Penalize $L^+_t, L^-_t$ differently in objective if desired

Overtime:
• Define $O_{jt}$ as hours of OT used on resource $j$ in period $t$
• Add $O_{jt}$ to $c_j$ in capacity constraint.
• Penalize $O_{jt}$ in objective if desired

Workforce Planning

Problem: determine most profitable production and hiring/firing policy over planning horizon.

Motivation for Study:
• hiring/firing vs. overtime vs. inventory build tradeoff
• iterative nature of optimization modeling.

Inputs:
• demand forecast (assume single product for simplicity)
• unit hour data
• labor content data
• capacity constraints
• hiring/firing costs
• overtime costs
• holding costs
• unit profit

Workforce Planning Notation

- $j$ = an index of workstation, $j = 1,...,n$
- $t$ = an index of period, $t = 1,...,T$
- $d_t^-$ = maximum demand in period $t$
- $d_t^+$ = minimum sales allowed in period $t$
- $a_j^o$ = unit hours on workstation $j$
- $b$ = number of man hours required to produce one unit
- $c_j^o$ = capacity of work center $j$ in period $t$
- $s$ = net profit from one unit
- $h$ = cost to hold one unit for one period $t$
- $l^r$ = cost of regular time in dollars/man-hour
- $l^o$ = cost of overtime in dollars/man-hour
- $e$ = cost to increase workforce by one man-hour
- $e'$ = cost to decrease workforce by one man-hour
Workforce Planning Notation (cont.)

- $X_t$ = amount produced in period $t$
- $S_t$ = amount sold in period $t$
- $I_t$ = inventory at end of $t$
- $W_t$ = workforce period $t$ in man-hours of regular time
- $H_t$ = increase (hires) in workforce from period $t-1$ to $t$ in man-hours
- $F_t$ = decrease (fires) in workforce from period $t-1$ to $t$ in man-hours
- $O_t$ = overtime in period $t$ in hours

Workforce Planning Formulation

$$\text{max} \sum_t \left( S_t - \alpha I_t - (W_t - O_t - eH_t - cF_t) \right)$$

subject to

- $2 \leq S_t \leq d_t$ for all $t$
- $0 \leq X_t \leq c_{X_t}$ for all $t$
- $I_t = I_{t-1} + X_t - S_t$ for all $t$
- $W_t = W_{t-1} + H_t - F_t$ for all $t$
- $\Delta X_t \leq W_t + O_t$ for all $t$
- $X_t, S_t, I_t, O_t, W_t, H_t, F_t \geq 0$ for all $t$

Workforce Planning Example

Problem Description

- 12 month planning horizon
- 168 hours per month
- 15 workers currently in system
- Regular time labor at $35 per hour
- Overtime labor at $52.50 per hour
- $2,500 to hire and train new worker
- $1,500 to lay off worker
- $2,500/168 \approx $14.88 = $15/hour
- $1,500/168 \approx $8.93 = $9/hour
- 12 hours labor per unit
- Demand assumed met ($S_t=d_t$, so $S_t$ variables are unnecessary)
Workforce Planning Example (cont.)

Solutions:
• "Chase" Solution: infeasible
• LP optimal Solution: layoff 9.5 workers
• Add constraint: \( F_t \geq 0 \)
  – results in 60 hours/worker/week of overtime
• Add constraint: \( O_t \leq 0.2W_t \)
  – Reasonable solution?

Conclusions

No single AP model is right for every situation
Simplicity promotes understanding
Linear programming is a useful AP tool
Robustness matters more than precision
Formulation and Solution are not separate activities.

Inventory Management

One’s work may be finished some day, but one’s education never.

– Alexandre Dumas
Hierarchical Planning – Roles of Inventory

Inventory is the Lifeblood of Manufacturing

Plays a role in almost all operations decisions
- shop floor control
- scheduling
- aggregate planning
- capacity planning, …

Links to most other major strategic decisions
- quality assurance
- product design
- facility design
- marketing
- organizational management, …

Managing inventory is close to managing the entire system...

Plan of Attack

Classification:
- raw materials
- work-in-process (WIP)
- finished goods inventory (FGI)
- spare parts

Justification:
- Why is inventory being held?
- benchmarking
Plan of Attack (cont.)

Structural Changes:
- major reorganization (e.g., eliminate stockpots, change purchasing contracts, alter product mix, focused factories, etc.)
- reconsider objectives (e.g., make-to-stock vs. make-to-order, capacity strategy, time-based-competition, etc.)

Modeling:
- What to model – identify key tradeoffs.
- How to model – EOQ, (Q,r), optimization, simulation, etc.

Raw Materials

Reasons for Inventory:
- batching (quantity discounts, purchasing capacity, …)
- safety stock (buffer against randomness in supply/production)
- obsolescence

Improvement Policies:
- Pareto analysis (focus on 20% of parts that represent 80% of $ value)
- ABC classification (stratify parts management)
- JIT deliveries (expensive and/or bulky items)
- vendor monitoring/management

Raw Materials (cont.)

Benchmarks:
- small C parts: 4-8 turns
- A,B parts: 12-25+ turns
- bulky parts: up to 50+ turns

Models:
- EOQ
- power-of-two
- service constrained optimization model
**Multiproduct EOQ Models**

Notation:
- \( N \) = total number of distinct part numbers in the system
- \( D_i \) = demand rate (units per year) for part \( i \)
- \( c_i \) = unit production cost of part \( i \)
- \( A_i \) = fixed cost to place an order for part \( i \)
- \( h_i \) = cost to hold one unit of part \( i \) for one year
- \( Q_i \) = the size of the order or lot size for part \( i \) (decision variable)

**Multiproduct EOQ Models (cont.)**

Cost-Based EOQ Model: For part \( i \),

\[
Q_i^* = \sqrt{\frac{2AD_i}{h_i}}
\]

but what is \( A \)?

Frequency Constrained EOQ Model:

\[
\begin{align*}
\min & \quad \text{inventory holding cost} \\
\text{subject to:} & \quad \frac{1}{N} \sum_{i=1}^{N} \frac{A_i D_i}{Q_i} \leq F
\end{align*}
\]

**Multiproduct EOQ Solution Approach**

Constraint Formulation:

\[
\begin{align*}
\min & \quad \frac{1}{N} \sum_{i=1}^{N} \frac{A_i D_i}{Q_i} \\
\text{subject to:} & \quad \frac{1}{N} \sum_{i=1}^{N} \frac{A_i D_i}{Q_i} \leq F
\end{align*}
\]

Cost Formulation:

\[
Y(Q) = \frac{1}{2} \left( \sum_{i=1}^{N} A_i Q_i \right) + \frac{1}{2} \sum_{i=1}^{N} \frac{A_i D_i}{Q_i}
\]
Multiproduct EOQ Solution Approach (cont.)

Cost Solution: Differentiate $Y(Q)$ with respect to $Q_i$, set equal to zero, and solve:

$$\frac{dY}{dQ_i} = \frac{AD_i}{2Q_i} = \frac{A}{Q_i}.$$  
No surprise - regular EOQ formula

Constraint Solution: For a given $A$ we can find $Q(A)$ using the above formula. The resulting average order frequency is:

$$F(A) = \frac{1}{N} \sum_{i=1}^{N} \frac{D_i}{Q_i(A)}.$$  
If $F(A) < F$ then penalty on order frequency is too high and should be decreased. If $F(A) > F$ then penalty is too low and needs to be increased.

Multiproduct EOQ Procedure – Constrained Case

Step (0) Establish a tolerance for satisfying the constraint (i.e., a sufficiently small number that represents “close enough” for the order frequency) and guess a value for $A$.

Step (1) Use $A$ in previous formula to compute $Q_i(A)$ for $i = 1, \ldots, N$.

Step (2) Compute the resulting order frequency:

$$F(A) = \frac{1}{N} \sum_{i=1}^{N} \frac{D_i}{Q_i(A)}.$$  
If $|F(A) - F| < \varepsilon$, STOP. $Q_i^* = Q_i(A)$, $i = 1, \ldots, N$. ELSE,  
If $F(A) < F$, decrease $A$  
If $F(A) > F$, increase $A$  
Go to Step (1).

Note: The increases and decreases in $A$ can be made by trial and error, or some more sophisticated search technique, such as interval bisection.

Multiproduct EOQ Example

Input Data:

<table>
<thead>
<tr>
<th>Part</th>
<th>$D_i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
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<td>100</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
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<tr>
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### Multiproduct EOQ Example (cont.)

#### Calculations:

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<tr>
<th>Iteration</th>
<th>A</th>
<th>Q1(A)</th>
<th>Q2(A)</th>
<th>Q3(A)</th>
<th>Q4(A)</th>
<th>F(A)</th>
<th>Inventory Investment</th>
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<td>10.00</td>
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<td>35.36</td>
<td>111.80</td>
<td>11.18</td>
<td>35.36</td>
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<td>12.25</td>
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<td>37.08</td>
<td>117.26</td>
<td>11.73</td>
<td>37.08</td>
<td>37.08</td>
<td>12.25</td>
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<tr>
<td>11</td>
<td>65.63</td>
<td>31.62</td>
<td>100.00</td>
<td>10.00</td>
<td>31.62</td>
<td>13.70</td>
<td>27.39</td>
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<tr>
<td>12</td>
<td>75.00</td>
<td>38.73</td>
<td>122.47</td>
<td>12.25</td>
<td>38.73</td>
<td>11.18</td>
<td>32.12</td>
</tr>
</tbody>
</table>

#### Powers-of-Two Adjustment

**Rounding Order Intervals:**

- \( T_1' = Q_1' / D_1 = 36.09 / 1000 = 0.03609 \text{ yrs} = 13.17 \text{ days} \)
- \( T_2' = Q_2' / D_2 = 114.14 / 1000 = 0.11414 \text{ yrs} = 41.66 \text{ days} \)
- \( T_3' = Q_3' / D_3 = 11.41 / 100 = 0.11414 \text{ yrs} = 41.66 \text{ days} \)
- \( T_4' = Q_4' / D_4 = 36.09 / 100 = 0.3609 \text{ yrs} = 131.73 \text{ days} \)

**Rounded Order Quantities:**

- \( Q_1' = D_1 \times T_1' / 365 = 100 \times 0.03609 \times 365 = 131.73 \)
- \( Q_2' = D_2 \times T_2' / 365 = 100 \times 0.11414 \times 365 = 394.64 \)
- \( Q_3' = D_3 \times T_3' / 365 = 100 \times 0.11414 \times 365 = 394.64 \)
- \( Q_4' = D_4 \times T_4' / 365 = 100 \times 0.3609 \times 365 = 131.73 \)

**Resulting Inventory and Order Frequency:** Optimal inventory investment is $3,126.53 and order frequency is 12. After rounding to nearest powers-of-two, we get:

- Inventory Investment: \( \sum \frac{Q_i}{2} = 3,243.64 \)
- Average Order Frequency: \( \frac{1}{4} \sum \frac{Q_i}{D_i} = 12.12 \)
Questions – Raw Materials

- Do you track vendor performance (i.e., as to variability)?
- Do you have a vendor certification program?
- Do your vendor contracts have provisions for varying quantities?
- Are purchasing procedures different for different part categories?
- Do you make use of JIT deliveries?
- Do you have excessive “wait to match” inventory? (May need more safety stock of inexpensive parts.)
- Do you have too many vendors?
- Is current order frequency rationalized?

Work-in-Process

Reasons for Inventory:

- queuing (variability)
- processing
- waiting to move (batching)
- moving
- waiting to match (synchronization)

Work-in-Process (cont.)

Improvement Policies:

- pull systems
- synchronization schemes
- lot splitting
- flow-oriented layout, floating work
- setup reduction
- reliability/maintainability upgrades
- focused factories
- improved yield/rework
- better scheduling
- judicious vending
Work-in-Process (cont.)

Benchmarks:
- coefficients of variation below one
- WIP below 10 times critical WIP
- relative benchmarks depend on position in supply chain

Models:
- queuing models
- simulation

Science Behind WIP Reduction

Cycle Time:
\[ CT = \left( \frac{c^2 + s^2}{2} \right) \frac{u}{1-u} + t_f \]

WIP:
\[ WIP = CT \times TH = \left( \frac{c^2 + s^2}{2} \right) \frac{u}{1-u} + u \]

Conclusion: CT and WIP can be reduced by reducing utilization, variability, or both.

Questions – WIP
- Are you using production leveling and due date negotiation to smooth releases?
- Do you have long, infrequent outages on machines?
- Do you have long setup times on highly utilized machines?
- Do you move product infrequently in large batches?
- Do some machines have utilizations in excess of 95%?
- Do you have significant yield/rework problems?
- Do you have significant waiting inventory at assembly stations (i.e., synchronization problems?)
Finished Goods Inventory

Reasons for Inventory:
• respond to variable customer demand
  • absorb variability in cycle times
  • build for seasonality
  • forecast errors

Improvement Policies:
• dynamic lead time quoting
  • cycle time reduction
  • cycle time variability reduction
  • late customization
  • balancing labor/inventory
  • improved forecasting

Finished Goods Inventory (cont.)

Benchmarks:
• seasonal products: 6-12 turns
  • make-to-order products: 30-50+ turns
  • make-to-stock products: 12-24 turns

Models:
• reorder point models
  • queueing models
  • simulation

Questions – FGI

• All the WIP questions apply here as well
• Are lead times negotiated dynamically?
• Have you exploited opportunities for late customization (e.g., bank stocks, product standardization, etc.)?
• Have you adequately considered variable labor (seasonal hiring, cross-trained workers, overtime)?
• Have you evaluated your forecasting procedures against past performance?
Spare Parts Inventory

Reasons for Inventory:
• customer service
• purchasing/production lead times
• batch replenishment

Improvement Policies:
• separate scheduled/unscheduled demand
• increase order frequency
• eliminate unnecessary safety stock
• differentiate parts with respect to fill rate/order frequency
• forecast life cycle effects on demand
• balance hierarchical inventories

Spare Parts Inventory (cont.)

Benchmarks:
• scheduled demand parts: 6-24 turns
• unscheduled demand parts: 1-12+ turns (highly variable!)
• Wharton survey

Models:
• (Q,r)
  • distribution requirements planning (DRP)
  • multi-echelon models

Multi-Product (Q,r) Systems

Many inventory systems (including most spare parts systems) involve multiple products (parts)

Products are not always separable because:
• average service is a function of all products
• cost of holding inventory is different for different products

Different formulations are possible, including:
• constraint formulation (usually more intuitive)
• cost formulation (easier to model, can be equivalent to constraint approach)
Model Inputs and Outputs

Costs
Order (A) or Backorder (B)
Stockout (K)
Holding (H)

Inputs (by part)
Cost (c)
Mean LT demand (θ)
Std Dev of LT demand (σ)

MODEL

Stocking Parameters
(by part)
Order Quantity (Q)
Reorder Point (r)

Performance Measures
(by part and for system)
Order Frequency (F)
Fill Rate (S)
Backorder Level (B)
Inventory Level (I)

Multi-Prod (Q,r) Systems – Constraint Formulations

Backorder model
min Inventory investment
subject to:
Average order frequency ≤ F
Average backorder level ≤ B

Fill rate model
min Inventory investment
subject to:
Average order frequency ≤ F
Average fill rate ≥ S

Two different ways to represent customer service.

Multi Product (Q,r) Notation

N = number of distinct parts in the system
D_i = expected demand per year for part i
D_n = Σ D_i = total demand
l_i = replenishment lead time (assumed constant) for part i
θ_i = expected demand during replenishment lead time for part i
σ_i = standard deviation of demand during replenishment lead time for part i
p(x) = pmf of demand during lead time for part i
Q_i(x) = cdf of demand during lead time for part i
e_i = unit cost of part i
h_i = annual unit holding cost for part i
A = fixed cost per order
b = annual unit backorder cost (all parts)
k = cost per stockout (all parts)
Multi-Product \((Q,r)\) Notation (cont.)

**Decision Variables:**
- \(Q_i\): order quantity for part \(i\)
- \(r_i\): reorder point for part \(i\)
- \(s_i\): safety stock implied by \(r_i\)

**Performance Measures:**
- \(F_i(Q_i,r_i)\): average order frequency for part \(i\)
- \(S_i(Q_i,r_i)\): average service level (fill rate) for part \(i\)
- \(B_i(Q_i,r_i)\): average backorder level for part \(i\)
- \(I_i(Q_i,r_i)\): average inventory level for part \(i\)

**Decision Variables:**
- \(Q\): quantity
- \(r\): reorder point

**Performance Measures:**
- \(F(Q,r)\): average order frequency
- \(S(Q,r)\): average service level (fill rate)
- \(B(Q,r)\): average backorder level
- \(I(Q,r)\): average inventory level

---

**Backorder Constraint Formulation**

**Verbal Formulation:**

\[ \text{min} \quad \text{Inventory investment} \]
\[ \text{subject to:} \]
- Average order frequency \( \leq F \)
- Total backorder level \( \leq B \)

**Mathematical Formulation:**

\[ \begin{align*}
\text{min} & \quad \sum_{i=1}^{N} F_i(Q_i,r_i) \\
\text{subject to:} & \quad \sum_{i=1}^{N} F_i(Q_i,r_i) \leq F \\
& \quad \sum_{i=1}^{N} B_i(Q_i,r_i) \leq B
\end{align*} \]

**Backorder Cost Formulation**

**Verbal Formulation:**

\[ \text{min} \quad \text{Ordering Cost + Backorder Cost + Holding Cost} \]

**Mathematical Formulation:**

\[ \begin{align*}
\text{min} & \quad \sum_{i=1}^{N} \left( AF_i(Q_i,r_i) + bB_i(Q_i,r_i) + h_i(I_i(Q_i,r_i)) \right) \\
\end{align*} \]
Fill Rate Constraint Formulation

Verbal Formulation:

\[
\min \text{ Inventory investment} \\
\text{subject to: } \text{Average order frequency} \leq F \\
\text{Average fill rate} \geq S
\]

Mathematical Formulation:

\[
\min \sum_{i=1}^{N} c_i I(Q_i, r_i) \\
\text{subject to: } \frac{1}{D_i} \sum_{i=1}^{N} F(Q_i, r_i) \leq F \\
\frac{1}{D_i} \sum_{i=1}^{N} S(Q_i, r_i) \geq S
\]

“Coupling” of \( Q \) and \( r \) makes this hard to solve.

Fill Rate Cost Formulation

Verbal Formulation:

\[
\min \text{ Ordering Cost + Stockout Cost + Holding Cost}
\]

Mathematical Formulation:

\[
\min \sum_{i=1}^{N} \left( A_i F(Q_i, r_i) + b_i (1 - S(Q_i, r_i)) + h_i I(Q_i, r_i) \right)
\]

Note: a stockout cost penalizes each order not filled from stock by \( k \) regardless of the duration of the stockout

“Coupling” of \( Q \) and \( r \) makes this hard to solve.

Relationship Between Cost and Constraint Formulations

Method:

1) Use cost model to find \( Q \) and \( r \), but keep track of average order frequency and fill rate using formulas from constraint model.

2) Vary order cost \( A \) until order frequency constraint is satisfied, then vary backorder cost \( b \) (stockout cost \( k \)) until backorder (fill rate) constraint is satisfied.

Problems:

- Even with cost model, these are often large-scale integer nonlinear optimization problems, which are hard.
- Because \( B(Q, r), S(Q, r), I(Q, r) \) depend on both \( Q \) and \( r \), solution will be “coupled”, so step (2) above won’t work without iteration between \( A \) and \( b \) (or \( k \)).
**Type I (Base Stock) Approximation for Backorder Model**

**Approximation:**
- replace $B(Q_i, r_i)$ with base stock formula for average backorder level, $B(r_i)$
- Note that this "decouples" $Q_i$ from $r_i$ because $F(Q_i, r_i) = D(Q_i)$ depends only on $Q_i$ and not $r_i$

**Resulting Model:**
\[
\min \sum_{i=1}^{N_t} \left( A_i \frac{D_i}{Q_i} + b_i (r_i + \frac{Q_i - 1}{2} + c_i - b_i - B(r_i)) \right)
\]

**Solution of Approximate Backorder Model**

Taking derivative with respect to $Q_i$ and solving yields:
\[
Q_i^* = \sqrt{\frac{2AD_i}{h_i}} \quad \text{EOQ formula again}
\]

Taking derivative with respect to $r_i$ and solving yields:
\[
G_i(r_i^*) = \frac{b_i}{h_i + b_i} \Rightarrow r_i^* = \theta_i + \frac{b_i}{h_i + b_i} \quad \text{base stock formula again}
\]

if $G_i$ is normal($\theta_i, \sigma_i$), where $G_i(z) = \Phi(z)$

**Using Approximate Cost Solution to Get a Solution to the Constraint Formulation**

1) Pick initial $A, b$ values.
2) Solve for $Q_i, r_i$ using:
\[
Q_i^* = \sqrt{\frac{2AD_i}{h_i}} \quad r_i^* = \theta_i + \frac{b_i}{h_i + b_i}
\]
3) Compute average order frequency and backorder level:
\[
\bar{T} = \sum_{i=1}^{N_t} \frac{D_i}{Q_i}, \quad \bar{B} = \sum_{i=1}^{N_t} B_i(Q_i, r_i)
\]
   Note: use exact formula for $B(Q_i, r_i)$ not approx.

4) Adjust $A$ until $\bar{T} = T$  
   Adjust $b$ until $\bar{B} = B$  
   Note: search can be automated with Solver in Excel.
Type I and II Approximation for Fill Rate Model

Approximation:
- Use EOQ to compute $Q_i$ as before
- Replace $B(Q_i, r_i)$ with $B(r_i)$ (Type I approx) in inventory cost term
- Replace $S(Q_i, r_i)$ with $1 - B(r_i)/Q_i$ (Type II approx) in stockout term

Resulting Model:

$$\min \sum \frac{A}{Q_i} + \frac{AD}{Q_i}B(r_i) + h_i \left( \frac{Q_i + 1}{2} + r_i - \theta_i + B(r_i) \right)$$

Note: we use this approximate cost function to compute $r_i$ only, not $Q_i$.

Solution of Approximate Fill Rate Model

EOQ formula for $Q_i$ yields:

$$Q_i^* = \sqrt{\frac{2AD}{h_i}}$$

Taking derivative with respect to $r_i$ and solving yields:

$$G(r_i^*) = \frac{kD_i}{hQ_i} + h_i \Rightarrow r_i^* = \hat{r}_i + \frac{h_i}{kD_i}$$

Note: modified version of basestock formula, which involves $Q_i$.

Using Approximate Cost Solution to Get a Solution to the Constraint Formulation

1) Pick initial $A$, $k$ values.
2) Solve for $Q_i$, $r_i$ using:

$$Q_i^* = \sqrt{\frac{2AD_i}{h_i}}$$

$$r_i^* = \hat{r}_i + \frac{h_i}{kD_i}$$

3) Compute average order frequency and fill rate using:

$$\bar{F} = \frac{1}{N} \sum \frac{D_i}{Q_i}$$

$$\bar{S} = \frac{1}{D_i} \sum i S(Q_i, r_i)$$

Note: use exact formula for $S(Q_i, r_i)$ not approx.

4) Adjust $A$ until

Adjust $\theta$ until

$$\bar{F} = F$$

$$\bar{S} = S$$

Note: search can be automated with Solver in Excel.
Multi-Product \((Q,r)\) Insights

- All other things being equal, an optimal solution will hold less inventory (i.e., smaller \(Q\) and \(r\)) for an expensive part than for an expensive one.
- Reduction in total inventory investment resulting from use of “optimized” solution instead of constant service (i.e., same fill rate for all parts) can be substantial.
- Aggregate service may not always be valid:
  -- could lead to undesirable impacts on some customers
  -- additional constraints (minimum stock or service) may be appropriate

Questions – Spare Parts Inventory

- Is scheduled demand handled separately from unscheduled demand?
- Are stocking rules sensitive to demand, replenishment lead time, and cost?
- Can you predict life-cycle demand better? Are you relying on historical usage only?
- Are your replenishment lead times accurate?
- Is excess distributed inventory returned from regional facilities to central warehouse?
- How are regional facility managers evaluated against inventory?
- Frequency of inspection?
- Are lateral transshipments between regional facilities being used effectively? Officially?

Multi-Echelon Inventory Systems

Questions:
- How much to stock?
- Where to stock it?
- How to coordinate levels?
Types of Multi-Echelon Systems

- **Level 1**: Serial System
- **Level 2**: General Arborescent System
- **Level 3**: Stocking Site

**Inventory Flow**

Two Echelon System

**Warehouse**
- evaluate with (Q,r) model
- compute stocking parameters and performance measures

**Facilities**
- evaluate with base stock model (ensures one-at-a-time demands at warehouse)
- consider delays due to stockouts at warehouse in replenishment lead times

Facility Notation

- \( D_m \): expected demand per year for part \( i \) at facility \( m \)
- \( \theta_m \): expected demand for part \( i \) during replenishment lead time at facility \( m \)
- \( \sigma_m \): standard deviation of demand for part \( i \) during replenishment leadtime at facility \( m \)
- \( d_m \): mean daily demand for part \( i \) at facility \( m \)
- \( \sigma_m (D) \): standard deviation of daily demand for part \( i \) at facility \( m \)
- \( p_m (t) \): pdf of demand during lead time for part \( i \) at facility \( m \)
- \( G_m (t) \): cdf of demand during lead time for part \( i \) at facility \( m \)
- \( W_i \): expected time an order for part \( i \) waits at warehouse due to backordering
- \( L_m \): lead time (including backorder delay) for an order of part \( i \) from facility \( m \)
- a random variable
### Warehouse Notation

- $N$ = total number of distinct parts in the system
- $M$ = number of facilities serviced by warehouse
- $D_i = \sum D_{im}$ = annual demand for part $i$ at the warehouse
- $\ell_i$ = replenishment lead time for part $i$ to the warehouse
- $\theta_i$ = expected demand for part $i$ during replenishment lead time at warehouse
- $\sigma_i$ = std dev of demand for part $i$ during replenishment lead time at warehouse
- $p_i(x)$ = pdf of demand during lead time for part $i$ at warehouse
- $G_i(x)$ = cdf of demand during lead time for part $i$ at warehouse
- $c_i$ = unit cost of for part $i$
- $h_i$ = unit holding cost for part $i$
- $b_i$ = unit backorder cost for part $i$
- $A$ = fixed setup cost

---

### Variables and Measures in Two Echelon Model

**Decision Variables:**
- $Q_i$ = order quantity for part $i$ at warehouse
- $x_i$ = reorder point for part $i$ at warehouse
- $r_{im}$ = reorder point for part $i$ at facility $m$
- $R_m$ = base stock level for part $i$ at facility $m$

**Performance Measures:**
- $F_i(Q_i, r_i) = \text{average order frequency for part } i \text{ at warehouse}$
- $S_i(Q_i, r_i) = \text{average service level} \text{ of fill rate for part } i \text{ at warehouse}$
- $B_i(Q_i, r_i) = \text{average backorder level} \text{ for part } i \text{ at warehouse}$
- $I_i(Q_i, r_i) = \text{average inventory level} \text{ for part } i \text{ at warehouse}$
- $S_{im}(R_{m}) = \text{average service level (fill rate) for part } i \text{ at facility } m$
- $B_{im}(R_{m}) = \text{average backorder level (fill rate) for part } i \text{ at facility } m$
- $I_{im}(R_{m}) = \text{average inventory level (fill rate) for part } i \text{ at facility } m$

---

### Facility Lead Times (mean)

**Delay due to backordering:**

$$ W_i = B_i(Q_i, r_i) / D_i \quad \text{by Little's law} $$

**Effective lead time for part $i$ to facility $m$:**

$$ E[L_{m}] = \ell_{m} + W_i $$

$$ \theta_{m} = D_{m}E[L_{m}] $$

*use this in place of $\theta$ in base stock model for facilities*
Facility Lead Times (std dev)

If \( y \) = delay for an order that encounters stockout, then:

\[
EL_{L_{im}} = \frac{S}{(1-S)}W
\]

\( y \) =

Variance of \( L_{im} \):

\[
Variance(L_{im}) = E[L_{im}^2] = (1-S)\{y^2 + E[L_{im}]^2\} - E[L_{im}]^2
\]

\[
\sigma_{L_{im}} = \sqrt{E[L_{im}^2]} = \sqrt{(1-S)\{y^2 + E[L_{im}]^2\} - E[L_{im}]^2}
\]

\( \sigma_{L_{im}} \) = we can use this in place of \( \sigma \) in normal base stock model for facilities

Two Echelon (Single Product) Example

\( D = 14 \) units per year (Poisson demand) at warehouse
\( s = 45 \) days
\( Q = 5 \)
\( r = 3 \)
\( D_m = 7 \) units per year at a facility
\( s_m = 1 \) day (warehouse to facility)

\( B(Q,r) = 0.0114 \)

\( S(Q,r) = 0.9721 \)

\( W = 365 \times 0.0114 = 0.296 \) days

\( \theta_m \) = \( \frac{D_m}{W} = \frac{7}{0.296} = 2.36 \)

Conclusion:

base stock level of 2 probably reasonable for facility.

Two Echelon Example (cont.)

Standard deviation of demand during replenishment lead time:

\[
\sigma(L_{im}) = \sqrt{E[L_{im}^2]} = \sqrt{0.9225 - 0.296 - 1.7474} = 0.296
\]

\( \sigma_{L_{im}} = \sqrt{E[L_{im}^2]} = \sqrt{0.9225 - 0.296 - 1.7474} = 0.1612 \)

Backorder level:

\( B(1) = 0.9225 \)

\( B(2) = 1 \)

computed from base stock model using \( \theta_m \) and \( \sigma_{L_{im}} \)

Conclusion: base stock level of 2 probably reasonable for facility.
Observations on Multi-Echelon Systems

- Service at central DC is a means to an ends (i.e., service at facilities).
- Service matters at locations that interface with customers:
  - fill rate (fraction of demands filled from stock)
  - average delay (expected wait for a part)
- Multi-echelon systems are hard to model/solve exactly, so we try to "decouple" levels.
  - Example: set fill rate at DC and compute expected delay at facilities, then search over DC service to minimize system cost.
- Structural changes are an option
  - (e.g., change number of DC's or facilities, allow cross-sharing, have suppliers deliver directly to outlets, etc.)