Electric Charge is a fundamental conserved property of some elementary particles. Electrical charges produce, electromagnetic fields and are influenced by electromagnetic fields. Electric charge of elementary particles determines their electromagnetic interaction. The electromagnetic force is the force that the electromagnetic field exerts on electrically charged particles. The electromagnetic force is one of the four fundamental forces of nature.

Some History
Thales of Miletus (around 600 BC) reported that charge could be accumulated by rubbing fur on various substances, such as amber. The Greeks noted that the charged amber could attract light objects such as hair. They also noted that if to rub amber for long enough, one can even get a spark to jump.

In 1600 William Gilbert in his tractate De Magnete coined the New Latin word electricus from ηλεκτρον (elektron), the Greek word for "amber", which gave rise to the English words "electric" and "electricity."

In 1873 James Clerk Maxwell in Treatise on Electricity and Magnetism formulated the system of equations that form the basis of classical theory of electromagnetic phenomena.

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Benjamin Franklin stated that a rubbed glass rod acquires the positive charge, and a rubbed amber rod acquires the negative charge.

Elementary particles

<table>
<thead>
<tr>
<th>Particle</th>
<th>Charge</th>
<th>Lifetime</th>
<th>Particle</th>
<th>Charge</th>
<th>Lifetime (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron ((e))</td>
<td>(-e)</td>
<td>Stable</td>
<td>Nutrino (muon) ((\nu_{\mu}))</td>
<td>0</td>
<td>Stable</td>
</tr>
<tr>
<td>Proton ((p))</td>
<td>(+e)</td>
<td>(10 \cdot 10^{35}) years</td>
<td>Pion ((\pi^+))</td>
<td>(+e)</td>
<td>(2.60 \cdot 10^{-8})</td>
</tr>
<tr>
<td>Neutron ((n))</td>
<td>0</td>
<td>Stable</td>
<td>Pion ((\pi^0))</td>
<td>0</td>
<td>(0.83 \cdot 10^{-16})</td>
</tr>
<tr>
<td>Muon ((\mu))</td>
<td>(-e)</td>
<td>(2.20 \cdot 10^{-6}) s</td>
<td>Tau ((\tau^-))</td>
<td>(-e)</td>
<td>(2.96 \cdot 10^{-13})</td>
</tr>
<tr>
<td>Nutrino ((electr)) ((\nu_e))</td>
<td>0</td>
<td>Stable</td>
<td>Nutrino (tau) ((\nu_{\tau}))</td>
<td>0</td>
<td>Stable</td>
</tr>
</tbody>
</table>

The charge is quantized: the electric charge of a macroscopic object is the sum of the electric charges of its constituent particles. The net electric charge of the neutral atoms and molecules is zero. The discrete nature of electric charge was proposed by Michael Faraday in electrolysis experiments, then proved by Robert Millikan in oil-drop experiment.

\[ Q = \pm Ne, \] where \(N\) is some integer.
**The law of conservation of charge**

**Charge conservation** is the principle that electric charge can neither be created nor destroyed and *the quantity of electric charge of the isolated physical system is always conserved*

Examples of charge conservation

The decay of the neutral pion on a time scale of about $10^{-16}$ s. The positive and negative pions have longer lifetimes of about $2.6 \cdot 10^{-8}$ s.

$$\pi^0 \rightarrow e^- + e^+ + \gamma,$$

$$\pi^{+/ -} \rightarrow \mu^{+/ -} + \nu$$

*Matter – antimatter interaction*

$$e^- + e^+ \rightarrow 2\gamma$$

electron antielectron $E$–$M$ waves

Energy $\geq 2m_e c^2$
Electron-positron pair production

When a photon energy is higher than $2m_e c^2$, one of the ways this photon interacts with matter is by producing an electron-positron pair.

\[ \text{Charge conservation in nuclear reactions} \quad ^{238}_{92}U \rightarrow ^{234}_{90}Th + ^4_2He \]

$Z/A -$ here $Z$ is the number of protons + neutrons, $A$ is the number of protons (electrons).

The atomic number $Z$ defines the nucleus of an element atom

\text{Disintegration of a neutron:} \quad n \rightarrow p^+ + e^- + \bar{\nu}

\text{Proton-proton collision:} \quad p^+ + p^+ \rightarrow n + p^+ + \pi^+$
**Quarks**

Quark is a generic particle that interact via the *strong force*. Quarks combined in specific ways form protons and neutrons, in each case taking exactly three quarks to make the composite particle in question. The notion of quarks was introduced independently by Murray Gell-Mann and Kazuhiko Nishijima (1961).

The key evidence for their existence came from a series of inelastic electron-nucleon scattering experiments conducted between 1967 and 1973 at the Stanford Linear Accelerator Center. Other theoretical and experimental advances of the 1970s confirmed this discovery, leading to the present standard model of elementary particle physics.

<table>
<thead>
<tr>
<th>Quark</th>
<th>Symbol</th>
<th>Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>$u$</td>
<td>$+\frac{2}{3}$</td>
</tr>
<tr>
<td>Down</td>
<td>$d$</td>
<td>$-\frac{1}{3}$</td>
</tr>
<tr>
<td>Charm</td>
<td>$c$</td>
<td>$+\frac{2}{3}$</td>
</tr>
<tr>
<td>Strange</td>
<td>$s$</td>
<td>$-\frac{1}{3}$</td>
</tr>
<tr>
<td>Top</td>
<td>$t$</td>
<td>$+\frac{2}{3}$</td>
</tr>
<tr>
<td>Bottom</td>
<td>$b$</td>
<td>$-\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Proton: $q = +\frac{2}{3} + \frac{2}{3} - \frac{1}{3}$

Neutron: $q = +\frac{2}{3} - \frac{1}{3} - \frac{1}{3}$

Omega ($\Omega$): $\Omega^- = -\frac{1}{3} - \frac{1}{3} - \frac{1}{3}$
Conductors and Insulators

- Conductors are materials in which electrical charges move freely
- Insulators are materials in which electrical charges cannot move
- Semiconductors are conducting materials in which the amount of freely moving electrical charges can be changed by external influence or by introduction of particular impurities.

Conducting substances: metals, semiconductors, superconductors, electrolytes, special alloys.

Coulomb’s Law (1785)

$$ F \sim \frac{|q_1||q_2|}{r^2} $$

$$ [F] \sim \left( \frac{\text{Charge}}{L} \right)^2 \quad \Leftrightarrow \quad N \sim \frac{C^2}{m^2} $$

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Electric Charge. Electric Fields

\[ F = k \frac{|q_1| |q_2|}{r^2} \]

\[ [F] = [k] \left[ \frac{\text{Charge}}{L} \right]^2, \quad \iff \quad N = [k] \frac{C^2}{m^2}, \quad [k] = \frac{N \cdot m^2}{C^2}, \]

\[ k = 8.99 \cdot 10^9 \frac{N \cdot m^2}{C^2} = \frac{1}{4\pi\varepsilon_0}, \]

\[ \varepsilon_0 = 8.859 \cdot 10^{-12} \frac{C^2}{N \cdot m^2} \]

**Coulomb’s Law**

\[ F_E = \frac{1}{4\pi\varepsilon_0} \frac{|q_1| |q_2|}{r^2} \]

**Law of Gravity**

\[ F_g = G \frac{m_1 m_2}{r^2} \]

**SI unit for electric charge**

\[ i = \Delta q / \Delta t \]

**SI unit of charge — coulomb (C)**

\[ i = 1A, \quad \Rightarrow \quad \Delta q = 1C, \quad \Delta t = 1s \]

\[ 1C = 1A \cdot s \]

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Millicoulomb $1mC = 10^{-3} C$, Microcoulomb $1\mu C = 10^{-6} C$, Nanocoulomb $1nC = 10^{-9} C$, Picocoulomb $1pC = 10^{-12} C$.

**The charge of electron** $e = 1.60 \cdot 10^{-19} C$

*Orientation of the electrostatic force*

$$F_{12} = F_{21} = \frac{1}{4\pi\varepsilon_0} \frac{|q_1||q_2|}{r^2}$$

$$\vec{F}_{21} = -\vec{F}_{12}.$$
Electric Charge. Electric Fields

Electric Force due to a System of Charges

\[ \vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} \]

\[ \vec{F}_{13} = \left[ \left( \vec{F}_{12} \right)_x + \left( \vec{F}_{13} \right)_x \right] \hat{i} + \left[ \left( \vec{F}_{12} \right)_y + \left( \vec{F}_{13} \right)_y \right] \hat{j} \]

\[ + \left[ \left( \vec{F}_{12} \right)_z + \left( \vec{F}_{13} \right)_z \right] \hat{k} \]

\[ \vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n} \]

\[ \vec{F}_{1,\text{net}} = \sum_{i=1}^{n} \vec{F}_{1i} \]
Comparison of electrostatic and gravitational forces in hydrogen atom

$$F_E = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}, \quad F_g = G \frac{m_e m_p}{r^2}, \quad r = r_B = 5.3 \cdot 10^{-11} m,$$

$$G = 6.7 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2}, \quad m_e = 9.11 \cdot 10^{-31} \text{ kg}, \quad m_p = 1.67 \cdot 10^{-27} \text{ kg},$$

$$F_E = \left(8.99 \cdot 10^9 \frac{N \cdot m^2}{C^2}\right) \frac{(1.6 \cdot 10^{-19} C)^2}{(5.3 \cdot 10^{-11} m)^2} = \frac{8.99 \cdot 2.56 \cdot 10^{38}}{28.09} \frac{N \cdot m^2}{kg^2} = 0.82 \cdot 10^{-7} N,$$

$$F_g = \left(6.7 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2}\right) \frac{(9.11 \cdot 10^{-31} \text{ kg})(1.67 \cdot 10^{-27} \text{ kg})}{(5.3 \cdot 10^{-11} m)^2} = \frac{6.7 \cdot 1.67 \cdot 10^{-11} \cdot 10^{-58}}{(5.3)^2 \cdot 10^{-22}} \frac{N \cdot m^2}{kg^2} = 3.63 \cdot 10^{-11} \cdot 10^{-36} N = 3.63 \cdot 10^{-47} N,$$

$$F_E = 8.2 \cdot 10^{-8} N \quad \quad \quad F_g = 3.63 \cdot 10^{-47} N \ll F_e$$
Sample Problem

Two charges $q_1 = 1.60\cdot10^{-19} \text{ C}$ and $q_2 = 3.20\cdot10^{-19} \text{ C}$ are fixed in place on the $x$–axis with a charge separation of $R = 0.0200 \text{ m}$. Charged particle $q_3 = -3.2\cdot10^{-19} \text{ C}$ is placed as it is shown in the figure. What is the net electrostatic force on the particle 1?

Solution

Let us introduce $q_1 = 1.60\cdot10^{-19} \text{ C} = q$, then $q_2 = 2q$, and $q_3 = -2q$. Orientation of the forces acting on the charge q is shown below

$$
\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13},
\vec{F}_{12} = -F_{12} \hat{i},
\vec{F}_{13} = (F_{13})_x \hat{i} + (F_{13})_y \hat{j},
$$

$$
\vec{F}_{1,net} = \left[ (F_{13})_x - F_{12} \right] \hat{i} + (F_{13})_y \hat{j}
$$
\[
\left( \vec{F}_{13} \right)_x = F_{13} \cos \theta, \quad \left( \vec{F}_{13} \right)_y = F_{13} \sin \theta,
\]

\[
\vec{F}_{1,\text{net}} = \left[ F_{13} \cos \theta - F_{12} \right] \hat{i} + (F_{13} \sin \theta) \hat{j},
\]

\[
F_{12} = \frac{1}{4\pi\varepsilon_0} \frac{|q_1||q_2|}{R^2} = \frac{1}{4\pi\varepsilon_0} \frac{|q||2q|}{R^2} = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2},
\]

\[
F_{13} = \frac{1}{4\pi\varepsilon_0} \frac{|q_1||q_3|}{(3R/4)^2} = \frac{1}{4\pi\varepsilon_0} \frac{|q||2q|}{(3R/4)^2} = \frac{1}{4\pi\varepsilon_0} \frac{16}{9} \frac{2q^2}{R^2},
\]

\[
\vec{F}_{1,\text{net}} = \left[ \frac{1}{4\pi\varepsilon_0} \frac{16}{9} \frac{2q^2}{R^2} \cos \theta - \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2} \right] \hat{i} + \left( \frac{1}{4\pi\varepsilon_0} \frac{16}{9} \frac{2q^2}{R^2} \sin \theta \right) \hat{j}
\]

\[
\vec{F}_{1,\text{net}} = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2} \left[ \left( \frac{16}{9} \cos(60^\circ) - 1 \right) \hat{i} + \frac{16}{9} \sin(60^\circ) \hat{j} \right],
\]

\[
\cos(60^\circ) = 0.5, \quad \sin(60^\circ) = 0.866,
\]
Electric Charge. Electric Fields

\[ \vec{F}_{1,\text{net}} = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2} \left\{ (0.89 - 1)\hat{i} + 1.54\hat{j} \right\} = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2} \left\{ -0.11\hat{i} + 1.54\hat{j} \right\} \]

Substituting values for \( R \), and \( q \) we get

\[ \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{R^2} = \left( 8.99 \cdot 10^9 \frac{N \cdot \text{m}}{\text{C}^2} \right) \frac{2 \left( 1.6 \cdot 10^{-19} \right)^2 \varepsilon_0^2}{(2 \cdot 10^{-2} \text{m})^2} = 11.51 \frac{10^9 \cdot 10^{-38}}{10^{-4}} N = 11.51 \cdot 10^{-25} N, \]

\[ \vec{F}_{1,\text{net}} = \left( 11.51 \cdot 10^{-25} N \right) \left[ -0.11\hat{i} + 1.54\hat{j} \right] = -\left( 1.27 \cdot 10^{-25} N \right)\hat{i} + \left( 17.7 \cdot 10^{-25} N \right)\hat{j}, \]

\[ \vec{F}_{1,\text{net}} = -\left( 1.3 \cdot 10^{-25} N \right)\hat{i} + \left( 1.8 \cdot 10^{-24} N \right)\hat{j}. \]

\[ \left( \vec{F}_{1,\text{net}} \right)_y / \left( \vec{F}_{1,\text{net}} \right)_x = \tan \alpha, \quad \alpha = \pi - \theta_{1,\text{net}}, \]

\[ \alpha = \arctan \left( \frac{1.8 \cdot 10^{-24} N}{1.3 \cdot 10^{-25} N} \right) = \arctan \left( \frac{1.8}{0.13} \right) = 85.87^\circ \approx 86^\circ, \]

\[ \theta_{1,\text{net}} = (\pi - \alpha), \quad \theta_{1,\text{net}} = 180^\circ - 86^\circ = 94^\circ, \quad \theta_{1,\text{net}} = 94^\circ \]
Sample Problem

Two identical small charged spheres, each having a mass of $3.0 \cdot 10^{-2} \text{ kg}$, hang in equilibrium as shown in the figure. The length of the suspension is equal to $L = 0.15 \text{ m}$ the angle $\theta = 5.0^\circ$. Find the magnitude of the charge on each sphere.

Solution

The free body diagram is show below.

Two similar charges are repelled and the magnitude of the electrostatic force $F_E$ is

$$F_E = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2d)^2}.$$ 

This force is balanced by the component of the tension force $T \sin \theta$. To find the magnitude of the tension force we have to analyze the condition of equilibrium.

$$\sum_j \vec{F}_j = 0.$$
Electric Charge. Electric Fields

\[ F_E \hat{i} - (T \sin \theta) \hat{i} + (T \cos \theta) \hat{j} - (mg) \hat{j} = 0, \]

\[ \left[ F_E - (T \sin \theta) \right] \hat{i} + \left[ (T \cos \theta) - mg \right] \hat{j} = 0, \quad \Rightarrow \]

\[ F_E - (T \sin \theta) = 0, \quad (T \cos \theta) - mg = 0, \]

\[ (T \cos \theta) = mg, \quad \Rightarrow \quad T = mg / \cos \theta, \]

\[ F_E = T \sin \theta, \quad \Rightarrow \quad F_E = mg \sin \theta / \cos \theta = mg \tan \theta, \]

\[ \frac{1}{4\pi \epsilon_0} \frac{q^2}{4d^2} = mg \tan \theta, \quad d = L \sin \theta, \quad \Rightarrow \quad \frac{1}{4\pi \epsilon_0} \frac{q^2}{4L^2 \sin^2 \theta} = mg \tan \theta, \]

\[ q^2 = \frac{mg \tan \theta}{\left(1 / (4\pi \epsilon_0)\right)} 4(L \sin \theta)^2, \quad \Rightarrow \quad q = 2L \sin \theta \sqrt{\frac{mg \tan \theta}{\left(1 / (4\pi \epsilon_0)\right)}}, \]

\[ q = 2L \sin \theta \sqrt{mg \tan \theta / \left(1 / (4\pi \epsilon_0)\right)}. \]
Electric Charge. Electric Fields

\[ q = 2(0.15 \text{ m}) \sin(5^\circ) \sqrt{\frac{(3.0 \cdot 10^2 \text{ kg})(9.8 \text{ m/s}^2) \tan(5^\circ)}{8.99 \cdot 10^9 \frac{N \cdot \text{m}^2}{C^2}}} \]

\[ = (0.3 \text{ m})0.087 \sqrt{\frac{3.98 \cdot 0.087 \cdot 10^2 \text{ kg} \cdot \text{m} \cdot C^2}{8.99 \cdot 10^9 \text{ s}^2 \cdot N \cdot \text{m}^3}} = (0.026 \text{ m}) \sqrt{0.28 \cdot 10^{-7} \frac{\text{kg} \cdot \text{C}^2 \cdot \text{m}^2}{\text{kg} \cdot \text{m} \cdot \text{m}}} \]

\[ = (0.026 \text{ m}) \sqrt{2.8 \cdot 10^{-8} \frac{C^2}{m^2}} = (0.026 \text{ m})1.67 \cdot 10^{-4} \frac{C}{\text{m}} = 4.3 \cdot 10^{-6} \text{ C}, \quad q = 4.3 \mu \text{C}. \]

Sample Problem

Two charges \( q_1 = +4q \) and \( q_2 = -q \) are fixed in place on the \( x \) – axis at the distance \( L \). Can a third charge be placed at some point on the \( x \) – axis so that it is in equilibrium? Is that equilibrium stable or unstable?
Solution

Suppose the third charge is \( q_0 > 0 \)

**Condition of equilibrium**

\[
\vec{F}_{0,\text{net}} = \vec{F}_{01} + \vec{F}_{02} = 0, \quad \vec{F}_{01} = -\vec{F}_{02}, \quad |\vec{F}_{01}| = |\vec{F}_{02}|.
\]
Electric Charge. Electric Fields

Case a

\[ F_{01} = \frac{1}{4\pi\varepsilon_0} \frac{|q_1||q_0|}{r_{01}^2} = \frac{1}{4\pi\varepsilon_0} \frac{4qq_0}{x^2}, \]

\[ F_{02} = \frac{1}{4\pi\varepsilon_0} \frac{|q_2||q_0|}{r_{02}^2} = \frac{1}{4\pi\varepsilon_0} \frac{qq_0}{(x-L)^2}. \]

\[ F_{01} = F_{02}, \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_0} \frac{4 \mathcal{A} q_0}{x^2} = \frac{1}{4\pi\varepsilon_0} \frac{\mathcal{A} q_0}{(x-L)^2} \quad \Rightarrow \quad \frac{4}{x^2} = \frac{1}{(x-L)^2}, \]

\[ \frac{4}{x^2} = \frac{1}{(x-L)^2}, \quad \Rightarrow \quad \frac{(x-L)^2}{x^2} = \frac{1}{4}, \quad \Rightarrow \quad \frac{(x-L)}{x} = \pm \frac{1}{2}. \]

1) \quad 2(x-L) = x, \quad 2x - 2L = x, \quad x = 2L, \quad x_1 = 2L

2) \quad 2(x-L) = -x, \quad 2x - 2L = -x, \quad 3x = 2L, \quad x_2 = 2L / 3L

\[ x = 2L \quad is \ \text{acceptable} \quad \text{Equilibrium is unstable} \]

\[ x = 2L / 3 < L \quad is \ \text{unacceptable} \]

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Case c

\[ \mathbf{F}_{01} q_0 \quad \mathbf{F}_{02} q_1 \quad q_2 \]

Equilibrium condition \( F_{01} = F_{02} \). It is evident that \( F_{01} > F_{02} \).

Let us prove it analytically

\[
F_{01} = F_{02}, \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_0} \frac{4q_1q_0}{x^2} = \frac{1}{4\pi\varepsilon_0} \frac{q_2q_0}{(x + L)^2}, \quad \Rightarrow \quad \frac{4}{x^2} = \frac{1}{(x + L)^2},
\]

\[
4 = \frac{x^2}{(x + L)^2}, \quad \Rightarrow \quad \pm 2 = \frac{x}{x + L},
\]

1) \( 2 = \frac{x}{x + L}, \quad 2(x + L) = x, \quad 2x + 2L = x, \quad x = -2L, \)

2) \( -2 = \frac{x}{x + L}, \quad -2(x + L) = x, \quad -2x - 2L = x, \quad -2L = 3x, \)

\( x \) is the distance and thus, has to be positive

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Two electrons (1 and 2) are on the x-axis and two ions of identical charge $-q$ are in fixed positions on the y-axis as shown in the figure. Electron 2 is free to move, while other particles are fixed in their places. For physically possible values of $q \leq 5e$, what are (a) the smallest, (b) the second smallest, and (c) third smallest values of $\theta$ for which electron 2 is held in place.

**Solution**

Let us find the sum of all the forces inserted on the electron 2 by other charges.
Condition for the electron 2 to stay in place is

\[ \mathbf{F}_{2,\text{net}} = 0, \quad \Rightarrow \quad \sum \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} = 0, \]

\[ \sum \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} = 0, \quad \Rightarrow \]

\[ (\mathbf{F}_{21})_x + (\mathbf{F}_{23})_x + (\mathbf{F}_{24})_x = 0, \quad \quad (\mathbf{F}_{21})_y + (\mathbf{F}_{23})_y + (\mathbf{F}_{24})_y = 0, \]

\[ (\mathbf{F}_{21})_x = F_{21}, \quad \quad (\mathbf{F}_{21})_y = 0, \]

\[ F_{21} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{R^2}, \quad F_{23} = \frac{1}{4\pi\varepsilon_0} \frac{eq}{(R / \cos \theta)^2}, \quad F_{24} = \frac{1}{4\pi\varepsilon_0} \frac{eq}{(R / \cos \theta)^2} \]

\[ (\mathbf{F}_{23})_x = -F_{23} \cos \theta, \quad (\mathbf{F}_{23})_y = -F_{23} \sin \theta, \]
\[
\left( \vec{F}_{24} \right)_x = -F_{24} \cos \theta, \quad \left( \vec{F}_{24} \right)_y = F_{23} \sin \theta,
\]

\[
\left( \vec{F}_{21} \right)_x + \left( \vec{F}_{23} \right)_x + \left( \vec{F}_{24} \right)_x = F_{21} - F_{23} \cos \theta - F_{24} \cos \theta,
\]

\[
\left( \vec{F}_{21} \right)_y + \left( \vec{F}_{23} \right)_y + \left( \vec{F}_{24} \right)_y = 0 - F_{23} \sin \theta + F_{24} \sin \theta,
\]

\[F_{23} = F_{24}, \quad \Rightarrow\]

\[
\left( \vec{F}_{21} \right)_x + \left( \vec{F}_{23} \right)_x + \left( \vec{F}_{24} \right)_x = F_{21} - 2F_{23} \cos \theta, \quad \left( \vec{F}_{21} \right)_y + \left( \vec{F}_{23} \right)_y + \left( \vec{F}_{24} \right)_y = 0,
\]

\[
\left( \vec{F}_{21} \right)_x + \left( \vec{F}_{23} \right)_x + \left( \vec{F}_{24} \right)_x = 0, \quad \Rightarrow \quad F_{21} - 2F_{23} \cos \theta = 0,
\]
\[ F_{21} - 2F_{23} \cos \theta = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{R^2} - 2 \frac{1}{4\pi \varepsilon_0} \frac{eq}{(R / \cos \theta)^2} \cos \theta, \]

\[ \frac{e}{4\pi \varepsilon_0} \left[ \frac{e}{R^2} - \frac{2q}{(R / \cos \theta)^2} \cos \theta \right] = 0, \quad \left[ \frac{e}{R^2} - \frac{2q \cos^2 \theta}{R^2} \cos \theta \right] = 0, \]

\[ \frac{1}{R^2} \left[ e - 2q \cos^3 \theta \right] = 0, \quad e - 2q \cos^3 \theta = 0, \quad e = 2q \cos^3 \theta, \]

\[ \frac{e}{q} = 2 \cos^3 \theta. \]

\[ q/e = \frac{1}{2 \cos^3 \theta}, \quad \text{if} \quad q \leq 5e, \]

\[ \frac{1}{2 \cos^3 \theta} \leq 5, \quad \Rightarrow \quad \frac{1}{10} \leq \cos^3 \theta, \]

\[ \theta < 60^\circ \]
\[ \cos^3 \theta \geq \frac{1}{10}, \quad \Rightarrow \quad \cos \theta \geq \frac{1}{\sqrt[3]{10}}, \quad \Rightarrow \quad \theta \leq 62.34^\circ \]

For the physically possible values it is reasonable to suppose that \( q = ne \), for \( n = 1 \ldots 5 \), then

\[ \theta_n = \arccos \frac{1}{\sqrt[3]{2n}}. \]

(a) The smallest value of angle is \( \theta_1 = 37.5^\circ \) (or 0.654 rad).

(b) The second smallest value of angle is \( \theta_2 = 50.95^\circ \) (or 0.889 rad).

(c) The third smallest value of angle is \( \theta_3 = 56.6^\circ \) (or 0.988 rad).
Two charged particles with $q_1 = q_2 = 3.20 \cdot 10^{-19} \text{ C}$ are on the $y$-axis at the distances $d = 17.0 \text{ cm}$ from the origin. Particle 3 of the charge $q_3 = 6.40 \cdot 10^{-19} \text{ C}$ is moved gradually along the $x$-axis from $x = 0$ to $x = 5.0 \text{ m}$. At what values of $x$ the magnitude of the net force on the particle 3 be (a) minimum and (b) maximum. What are these maximum and minimum values?

**Solution**

\[
\cos \varphi = \frac{x}{\sqrt{x^2 + d^2}}
\]
Electric Charge

\[ \vec{F}_{3,\text{net}} = \vec{F}_{31} + \vec{F}_{32} = \left[ (\vec{F}_{31})_x + (\vec{F}_{32})_x \right] \hat{i} + \left[ (\vec{F}_{31})_y + (\vec{F}_{32})_y \right] \hat{j}, \]

\[ F_{31} = F_{32} = \frac{1}{4\pi\varepsilon_0} \frac{qq_3}{(x^2 + d^2)^2}, \]

\[ q_1 = q_2 = q = 2e \quad q_3 = 4e. \]

\[ \left( \vec{F}_{31} \right)_y = -F_{31} \sin \varphi, \quad \left( \vec{F}_{32} \right)_y = F_{32} \sin \varphi, \quad \Rightarrow \quad \left[ (\vec{F}_{31})_y + (\vec{F}_{32})_y \right] = 0, \]

\[ \left( \vec{F}_{31} \right)_x = \left( \vec{F}_{32} \right)_x = F_{31} \cos \varphi, \quad \vec{F}_{3,\text{net}} = (2F_{32} \cos \varphi) \hat{i}. \]

\[ F_{3,\text{net}} = \frac{1}{4\pi\varepsilon_0} \frac{2e(4e)}{(x^2 + d^2)^3} \frac{x}{\sqrt{x^2 + d^2}} = \frac{4e^2}{\pi\varepsilon_0} \frac{x}{\left( x^2 + d^2 \right)^{3/2}}, \]
Electric Charge

\[ F_{3,\text{net}} = F_{3,\text{net}}(x) \]

\[
\frac{dF_{3,\text{net}}}{dx} = 0, \quad \frac{d^2F_{3,\text{net}}}{dx^2} > 0 \quad \text{or} \quad \frac{d^2F_{3,\text{net}}}{dx^2} < 0
\]

\[
\frac{d^2F_{3,\text{net}}}{dx^2} > 0 \quad \leftrightarrow \quad \text{min}, \quad \frac{d^2F_{3,\text{net}}}{dx^2} < 0 \quad \leftrightarrow \quad \text{max}
\]

\[
\frac{dF_{3,\text{net}}}{dx} = \frac{d}{dx} \frac{4e^2}{\pi \varepsilon_0} \left( x^2 + d^2 \right)^{3/2} = \frac{4e^2}{\pi \varepsilon_0} \frac{d}{dx} \left( x^2 + d^2 \right)^{3/2} = \frac{4e^2}{\pi \varepsilon_0} \frac{d}{dx} \left[ x \left( x^2 + d^2 \right)^{-3/2} \right]
\]

\[
\frac{d}{dx} \left\{ x \left( x^2 + d^2 \right)^{-3/2} \right\} = x \left( x^2 + d^2 \right)^{-3/2} + x \frac{d}{dx} \left( x^2 + d^2 \right)^{-3/2} = x \left( x^2 + d^2 \right)^{-3/2} + \frac{3}{2} x \left( x^2 + d^2 \right)^{-5/2}
\]

\[
= \frac{1}{\left( x^2 + d^2 \right)^{3/2}} - \frac{3x^2}{\left( x^2 + d^2 \right)^{5/2}} = \frac{\left( x^2 + d^2 \right)-3x^2}{\left( x^2 + d^2 \right)^{5/2}} = \frac{d^2 - 2x^2}{\left( x^2 + d^2 \right)^{5/2}}
\]
\[
\frac{dF_{3,\text{net}}}{dx} = 0, \quad \Rightarrow \quad \frac{d^2 - 2x^2}{(x^2 + d^2)^{5/2}} = 0, \quad \Rightarrow \quad d^2 - 2x^2 = 0,
\]

\[
d^2 - 2x^2 = 0, \quad \Rightarrow \quad x_{1,2} = \pm \frac{d}{\sqrt{2}}.
\]

\[
\frac{d^2F_{3,\text{net}}}{dx^2} = \frac{4e^2}{\pi\varepsilon_0} \frac{d}{dx} \frac{d^2 - 2x^2}{(x^2 + d^2)^{5/2}} = \frac{4e^2}{\pi\varepsilon_0} \frac{d}{dx} \left\{ (d^2 - 2x^2) \left( x^2 + d^2 \right)^{-5/2} \right\} = \frac{4e^2}{\pi\varepsilon_0} \left\{ x^2 + d^2 \right\}^{-5/2} \frac{d}{dx} \left( d^2 - 2x^2 \right)
\]

\[
+ (d^2 - 2x^2) \frac{d}{dx} \left( x^2 + d^2 \right)^{-5/2} \right\} = \frac{4e^2}{\pi\varepsilon_0} \left\{ -4x \left( x^2 + d^2 \right)^{-5/2} + (d^2 - 2x^2) \left( -\frac{5}{2} \right) \left( x^2 + d^2 \right)^{-7/2} (2x) \right\}
\]

\[
= -\frac{4e^2}{\pi\varepsilon_0} \left\{ 4x \left( x^2 + d^2 \right)^{-5/2} + 5x (d^2 - 2x^2) \left( x^2 + d^2 \right)^{-7/2} \right\} = -\frac{4e^2}{\pi\varepsilon_0} \frac{4x \left( x^2 + d^2 \right) + 5x (d^2 - 2x^2)}{(x^2 + d^2)^{7/2}}
\]
\[
\frac{d^2 F_{3,\text{net}}}{dx^2} = -\frac{4e^2}{\pi \varepsilon_0} \frac{4x(x^2 + d^2) + 5x(d^2 - 2x^2)}{(x^2 + d^2)^{7/2}} = -\frac{4e^2}{\pi \varepsilon_0} \frac{4x^3 + 4xd^2 + 5xd^2 - 10x^3}{(x^2 + d^2)^{7/2}}
\]

\[
= -\frac{4e^2}{\pi \varepsilon_0} \frac{9xd^2 - 6x^3}{(x^2 + d^2)^{7/2}} = -\frac{4e^2}{\pi \varepsilon_0} \frac{3x(3d^2 - 2x^2)}{(x^2 + d^2)^{7/2}},
\]

\[
\frac{d^2 F_{3,\text{net}}}{dx^2} = -\frac{4e^2}{\pi \varepsilon_0} \frac{3x(3d^2 - 2x^2)}{(x^2 + d^2)^{7/2}}.<br
\]

\[
x = \frac{d}{\sqrt{2}}, \quad \Rightarrow \quad \frac{d^2 F_{3,\text{net}}}{dx^2} = -\frac{4e^2}{\pi \varepsilon_0} \frac{3d \left(3d^2 - \frac{2d^2}{\sqrt{2}}\right)}{\sqrt{2} (x^2 + d^2)^{7/2}} = -\frac{4e^2}{\pi \varepsilon_0} \frac{3d(2d^2)}{\sqrt{2} (x^2 + d^2)^{7/2}} < 0,
\]

\[
x = \frac{d}{\sqrt{2}}, \quad \Leftrightarrow \quad \text{corresponds to the maximum value of } F_{3,\text{net}},
\]
\[ x = - \frac{d}{\sqrt{2}}, \quad \Rightarrow \quad \frac{d^2 F_{3,\text{net}}}{dx^2} = -\frac{4e^2}{\pi \varepsilon_0} \left( -\frac{d}{\sqrt{2}} \right) \frac{3d^2 - \frac{2d^2}{x^2}}{\left( x^2 + d^2 \right)^{7/2}} = \frac{4e^2}{\pi \varepsilon_0} \frac{3d^2}{\sqrt{2} (x^2 + d^2)^{7/2}} > 0, \]

\[ x = - \frac{d}{\sqrt{2}}, \quad \Leftrightarrow \quad \text{corresponds to the minimum value of} \quad F_{3,\text{net}}. \]

\[ F_{3,\text{net}} \bigg|_{x = -d/\sqrt{2}} = \frac{4e^2}{\pi \varepsilon_0} \frac{d}{\sqrt{2}} \left( d^2 / 2 + d^2 \right)^{3/2} = \frac{4e^2}{\pi \varepsilon_0} \frac{d^2}{\sqrt{2} \left( 1/2 + 1 \right)^{3/2}} = \frac{4e^2}{\pi \varepsilon_0} \frac{1}{d^2 \sqrt{2} \left( 3/2 \right)^{3/2}} \]

\[ = \frac{4e^2}{\pi \varepsilon_0} \frac{2\sqrt{2}}{d^2 3 \sqrt{2} \sqrt{3}} = \frac{8}{\pi \varepsilon_0 3 \sqrt{3}} \frac{e^2}{d^2}, \]

\[ F_{3,\text{net}} \bigg|_{x = -d/\sqrt{2}} = \frac{8}{\pi \varepsilon_0 3 \sqrt{3}} \frac{e^2}{d^2}. \]
The minimum and maximum values of the magnitude of the net force.

\[
F_{3,\text{net}} = \frac{4e^2}{\pi \varepsilon_0} \left| \frac{x}{(x^2 + d^2)^{3/2}} \right|
\]

Thus, the minimum value of the force magnitude is \( F_{3,\text{net}} = 0 \).

The maximum values of the force magnitude is \( (F_{3,\text{net}})_{ax} = \frac{8}{\pi \varepsilon_0 3\sqrt{3}} \frac{e^2}{d^2} \).