1 Page 36, #41:

Write \( \tan \theta \) in terms of \( \sin \theta \).

1.1 Approach I

This is the most elegant and direct approach!

Fundamental Identity (FI):

\[
\tan = \frac{\sin}{\cos} \tag{1}
\]

FI:

\[
\sin^2 + \cos^2 = 1 \rightarrow \cos^2 = 1 - \sin^2 \rightarrow \cos = \sqrt{1 - \sin^2} \tag{2}
\]

Substitute eq. 2 into eq. 1:

\[
\frac{\sin}{\sqrt{1 - \sin^2}} \tag{3}
\]

1.2 Approach II

FI: \( \tan^2 + 1 = \sec^2 \):

\[
\tan^2 = \sec^2 - 1 \rightarrow \tan = \sqrt{\sec^2 - 1} \tag{4}
\]

FI: \( \cos = 1/\sec \rightarrow \sec = 1/\cos \). Substituting in to eq. 10:

\[
\sqrt{\sec^2 - 1} = \sqrt{\left(\frac{1}{\cos}\right)^2 - 1} = \sqrt{\frac{1}{\cos^2} - 1} \tag{5}
\]

FI: \( \sin^2 + \cos^2 = 1 \rightarrow \cos^2 = 1 - \sin^2 \). Substituting into eq. 5:

\[
\sqrt{\frac{1}{\cos^2} - 1} = \sqrt{\frac{1}{1 - \sin^2} - 1} \tag{6}
\]

Done! However, the book simplifies the radical, so let’s do some algebra.

\[
\sqrt{\frac{1}{1 - \sin^2} - 1} \tag{7}
\]

We need to have a common denominator of \( 1 - \sin^2 \), so let’s rewrite:
\[ \sqrt{\frac{1}{1-\sin^2} - \frac{1-\sin^2}{1-\sin^2}} \]  

(8)

Common denominator, so can combine and simplify:

\[ \sqrt{\frac{1-1+\sin^2}{1-\sin^2}} = \sqrt{\frac{\sin^2}{1-\sin^2}} = \frac{\sin}{\sqrt{1-\sin^2}} \]  

(9)

1.3 Approach III

FI: \( \tan = 1/\cot \).

Can we find some relationship between \( \cot \) and \( \sin \)?

Yes, between \( \csc \) and \( \sin \):

FI: \( 1 + \cot^2 = \csc^2 \) and

FI: \( \sin = 1/\csc \)

So, let’s bring everything together:

\[ \tan = \frac{1}{\cot} \]  

(10)

\[ 1 + \cot^2 = \csc^2 \rightarrow \cot = \sqrt{\csc^2 - 1} \]  

(11)

\[ \sin = \frac{1}{\csc} \rightarrow \csc = \frac{1}{\sin} \]  

(12)

Substituting eq. 11 into eq. 10:

\[ \frac{1}{\cot} = \frac{1}{\sqrt{\csc^2 - 1}} \]  

(13)

Substituting in eq. 12:

\[ \frac{1}{\sqrt{\csc^2 - 1}} = \frac{1}{\sqrt{\sin^2 - 1}} \]  

(14)

DONE!

Now simplify the radical. Let’s just take what’s underneath the radical:

\[ \frac{1}{\sqrt{\sin^2 - 1}} \]  

(15)

\[ \frac{1}{\sin^2} - 1 = \frac{1}{\sin^2} - \frac{\sin^2}{\sin^2} \]  

(16)

We now have a common denominator, so combine:

\[ \frac{1-\sin^2}{\sin^2} \]  

(17)

Bring back under the radical and simplify:

\[ \sqrt{\frac{1-\sin^2}{\sin^2}} = \frac{\sqrt{1-\sin^2}}{\sqrt{\sin^2}} = \frac{\sqrt{1-\sin^2}}{\sin} \]  

(18)
Substitute back in to the original equation:

\[
\frac{1}{\sqrt{\sin^2 - 1}} \rightarrow \frac{1}{\sqrt{1 - \sin^2}}
\]  \hspace{1cm} (19)

Flip the denominator and multiply:

\[
\frac{1}{\sin} \left( \frac{\sin}{\sqrt{1 - \sin^2}} \right) = \frac{\sin}{\sqrt{1 - \sin^2}}
\]  \hspace{1cm} (20)

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tan\(\theta\) in terms of sec\(\theta\):

Fl: \(\tan^2 + 1 = \sec^2\)

\[
\therefore \tan = \sqrt{\sec^2 - 1}
\]  \hspace{1cm} (21)

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sec\(\theta\) in terms of cot\(\theta\):

Fl: \(\tan^2 + 1 = \sec^2\)

Fl: \(\tan = 1/\cot\)

Therefore:

\[
\sec^2 = \tan^2 + 1 = \frac{1}{\cot^2} + 1
\]  \hspace{1cm} (22)

\[
\therefore \sec = \sqrt{\frac{1}{\cot^2} + 1}
\]  \hspace{1cm} (23)

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Write sec\(\theta\) in terms of sin\(\theta\):

Fl:

\[
\cos = 1/\sec \rightarrow \sec = 1/\cos
\]  \hspace{1cm} (24)

Fl:

\[
\sin^2 + \cos^2 = 1 \rightarrow \cos^2 = 1 - \sin^2 \rightarrow \cos = \sqrt{1 - \sin^2}
\]  \hspace{1cm} (25)

Substitute eq. 25 into eq. 24:

\[
\sec = \frac{1}{\cos} = \frac{1}{\sqrt{1 - \sin^2}}
\]  \hspace{1cm} (26)
Write cotθ in terms of cscθ:
FI: $1 + cot^2 = csc^2 \rightarrow cot^2 = csc^2 - 1$

\[ \therefore cot = \sqrt{csc^2 - 1} \quad (27) \]