Do problems:


from "Linear System Theory and Design" by Chi–Tsong Chen, published by Holt, Rinehart, and Winston, 1984 -- see attached scan.
2-12 Consider Table 2-1. Suppose the representations of \( b, e_1, e_2, e_3 \), and \( e_4 \) with respect to the basis \( \{e_1, e_2, e_3\} \) are known. Use Equation (2-20) to derive the representations of \( b, e_1, e_2, e_3 \), and \( e_4 \) with respect to the basis \( \{e_1, e_2\} \).

2-13 Show that similar matrices have the same characteristic polynomial, and consequently, the same set of eigenvalues. \([HINT: \det(AB) = \det A \det B]\)

2-14 Find the \( P \) matrix in Example 3, Section 2-4, and verify \( A = PAP^{-1} \).

2-15 Given

\[
\begin{align*}
A &= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & b &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} & \tilde{b} &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

what are the representations of \( A \) with respect to the basis \( \{b, Ab, A^2b, A^3b\} \) and the basis \( \{b, Ab, A^2b, A^3b\} \)? (Note that the representations are the same!)

2-16 What are the ranks and nullities of the following matrices?

\[
\begin{align*}
A_1 &= \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & -3 \\ 1 & 3 & 0 \end{bmatrix} & A_2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & A_3 &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 4 & 5 & 0 & 0 \end{bmatrix}
\end{align*}
\]

2-17 Find the bases of the range spaces and the null spaces of the matrices given in Problem 2-16.

2-18 Are the matrices

\[
\begin{bmatrix} s^3 + s^2 & s^2 + 1 \\ s & 1 \end{bmatrix} \quad \begin{bmatrix} s^2 + 1 \\ s^2 \end{bmatrix}
\]

nonsingular in the field of rational functions with real coefficients \( \mathbb{R}(s) \)? For every \( s \) in \( \mathbb{C} \), the matrices become numerical matrices with elements in \( \mathbb{C} \). For every \( s \) in \( \mathbb{C} \), are the matrices nonsingular in the field of complex numbers \( \mathbb{C} \)?

2-19 Does there exist a solution for the following linear equations?

\[
\begin{bmatrix} 3 & 3 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}
\]

If so, find one.

2-20 Consider the set of linear equations

\[
x(n) = A^4 x(0) + A^{n-1} b_0(0) + A^{n-2} b_0(1) + \cdots + A b_0(n-2) + b_0(n-1)
\]

where \( A \) is an \( n \times n \) constant matrix and \( b \) is an \( n \times 1 \) column vector. Given any \( x(n) \) and \( x(0) \), under what conditions on \( A \) and \( b \) will there exist \( u(0), u(1), \ldots, u(n-1) \) satisfying the equation? \([HINT: \text{Write the equation in the form}]

\[
x(n) - A^4 x(0) = \begin{bmatrix} b & A b & \cdots & A^{n-2} b \\ u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}
\]

2-21 Find the Jordan-canonical-form representations of the following matrices:

\[
A_1 = \begin{bmatrix}
1 & 4 & 10 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -4 & -3
\end{bmatrix}, \\
A_3 = \begin{bmatrix}
0 & 4 & 3 \\
0 & -150 & -120 \\
0 & 200 & 160
\end{bmatrix}, \quad A_4 = \begin{bmatrix}
0 & 4 & 3 \\
0 & 20 & 16 \\
0 & -25 & -20
\end{bmatrix}, \\
A_5 = \begin{bmatrix}
\frac{2}{5} & \frac{21}{5} & 14 \\
-\frac{5}{2} & -\frac{3}{2} & -2 \\
-\frac{1}{2} & -\frac{3}{2} & -2
\end{bmatrix}, \quad A_6 = \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

2-22 Let \( \lambda_i \) for \( i = 1, 2, \ldots, n \) be the eigenvalues of an \( n \times n \) matrix \( A \). Show that

\[
\det A = \prod_{i=1}^{n} \lambda_i
\]

2-23 Prove that a square matrix is nonsingular if and only if there is no zero eigenvalue.

2-24 Under what condition will \( AB = AC \) imply \( B = C \)? (\( A \) is assumed to be a square matrix.)

2-25 Show that the Vandermonde determinant

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{vmatrix}
\]

is equal to \( \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \).

2-26 Consider the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1
\end{bmatrix}
\]

Show that the characteristic polynomial of \( A \) is

\[
\Delta(\lambda) \triangleq \det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_{n-1} \lambda + \alpha_n
\]

If \( \lambda_1 \) is an eigenvalue of \( A \) (that is, \( \Delta(\lambda_1) = 0 \)), show that \( \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \end{bmatrix} \) is an eigenvector associated with \( \lambda_1 \). (The matrix \( A \) is called the companion matrix of the polynomial \( \Delta(\lambda) \). It is said to be in the Frobenius form in the numerical analysis literature.)
2-27 Consider the matrix shown in Problem 2-26. Suppose that \( \lambda_1 \) is an eigenvalue of the matrix with multiplicity \( k \); that is, \( \Delta(\lambda) \) contains \( (\lambda - \lambda_1)^k \) as a factor. Verify that the following \( k \) vectors,

\[
\begin{bmatrix}
1 \\
\lambda_1 \\
\lambda_1^2 \\
\vdots \\
\lambda_1^{n-2} \\
(n-1)\lambda_1^{n-3} \\
\frac{(n-1)}{2} \lambda_1^{n-4} \\
\frac{(n-1)}{3} \lambda_1^{n-5} \\
\vdots \\
\frac{(n-1)(n-2)}{i(i-1)} \lambda_1^{n-i-1} \\
\end{bmatrix}
\]

where

\[
\delta_i = \frac{(n-1)(n-2) \cdots (n-i)}{1 \cdot 2 \cdot 3 \cdots i} \quad i \geq 1
\]

are generalized eigenvectors of \( A \) associated with \( \lambda_1 \).

2-28 Show that the matrix \( A \) in Problem 2-26 is nonsingular if and only if \( a_n \neq 0 \). Verify that its inverse is given by

\[
A^{-1} = \begin{bmatrix}
-a_{n-1}/a_n & -a_{n-2}/a_n & \cdots & -a_1/a_n & -1/a_n \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix}
\]

2-29 Show that the determinant of the \( m \times m \) matrix

\[
\begin{bmatrix}
s^m & -1 & 0 & \cdots & 0 & 0 \\
0 & s^{m-1} & -1 & \cdots & 0 & 0 \\
0 & 0 & s^{m-2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s^2 & -1 \\
\beta_m(s) & \beta_{m-1}(s) & \beta_{m-2}(s) & \cdots & \beta_2(s) & s^1 + \beta_1(s) \\
\end{bmatrix}
\]

is equal to

\[s^n + \beta_1(s) s^{n-k_1} + \beta_2(s) s^{n-k_2} + \cdots + \beta_n(s)\]

where \( n = k_1 + k_2 + \cdots + k_m \) and \( \beta_i(s) \) are arbitrary polynomials.

2-30 Show that the characteristic polynomial of the matrix

\[
\begin{bmatrix}
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
-a_{21,m_1} & -a_{21,m-1} & \cdots & -a_{21,2} & -a_{21,1} & \cdots & -a_{21,2} & -a_{21,1} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-a_{21,m} & -a_{21,m-1} & \cdots & -a_{21,2} & -a_{21,1} & \cdots & -a_{21,2} & -a_{21,1} \\
\end{bmatrix}
\]
is given by
\[
\det \begin{bmatrix}
\Delta_{11}(s) & \Delta_{12}(s) \\
\Delta_{21}(s) & \Delta_{22}(s)
\end{bmatrix} = \Delta_{11}(s)\Delta_{22}(s) - \Delta_{12}(s)\Delta_{21}(s)
\]
where
\[
\begin{aligned}
\Delta_{i1}(s) &= s^n + a_{i1} s^{n-1} + \cdots + a_{i(n-1)} s + a_{in} \\
\Delta_{i2}(s) &= a_{i1} s^{n-1} + \cdots + a_{i(n-1)} s + a_{ij}
\end{aligned}
\]
Note that the submatrices on the diagonal are of the companion form (see Problem 2.26); the submatrices not on the diagonal are all zeros except the last row.

**2-31** Find the characteristic polynomials and the minimal polynomials of the following matrices:
\[
\begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
0 & 0 & \lambda_1 & 1 \\
0 & 0 & 0 & \lambda_2
\end{bmatrix},
\begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_1
\end{bmatrix},
\begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_1 & 1 \\
0 & 0 & 0 & \lambda_1
\end{bmatrix},
\begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_1
\end{bmatrix}
\]
What are the multiplicities and indices? What are their geometric multiplicities?

**2-32** Show that if \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( x \), then \( f(\lambda) \) is an eigenvalue of \( f(A) \) with the same eigenvector \( x \).

**2-33** Repeat the problems in Examples 2 and 3 of Section 2.7 by choosing, respectively, \( g(\lambda) = \lambda \alpha_0 + \alpha_1 (\lambda - 1) \) and \( g(\lambda) = \alpha_0 (\lambda - 1) + \alpha_1 (\lambda - 1)^2 (\lambda - 2) + \alpha_2 (\lambda - 2) \).

**2-34** Given
\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
Find \( A^{10}, A^{103}, \) and \( e^{\lambda x} \).

**2-35** Compute \( e^{\lambda x} \) for the matrices
\[
\begin{bmatrix}
1 & 4 & 10 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix},
\begin{bmatrix}
0 & 4 & 3 \\
0 & -150 & -120 \\
0 & 200 & 160
\end{bmatrix}
\]
by using Definition 2.16 and by using the Jordan-form representation.

**2-36** Show that functions of the same matrix commute, that is,
\[
f(A)g(A) = g(A)f(A)
\]
Consequently, we have \( A e^{\lambda x} = e^{\lambda x} A \).

**2-37** Let
\[
C = \begin{bmatrix}
\lambda_2 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]
Find a matrix \( B \) such that \( e^B = C \). Show that if \( \lambda_i = 0 \) for some \( i \) then the matrix \( B \) does
not exist. Let

\[ C = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \]

Find a matrix B such that \( \phi = C \). \([\text{Hint: Let } f(\lambda) = \log \lambda \text{ and use (2.69).} \] Is it true that for any nonsingular matrix C, there exists a matrix B such that \( \phi = C \)?

2-38 Let \( A \) be an \( n \times n \) matrix. Show by using the Cayley–Hamilton theorem that any \( A^k \) with \( k \geq n \) can be written as a linear combination of \( \{I, A, \ldots, A^{n-1}\} \). If the degree of the minimal polynomial of \( A \) is known, what modification can you make?

2-39 Define

\[(sI - A)^{-1} \triangleq \frac{1}{\Delta(s)} [R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2} s + R_{n-1}] \]

where \( \Delta(s) \triangleq \det(sI - A) \triangleq s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n \) and \( R_0, R_1, \ldots, R_{n-1} \) are constant matrices. This definition is valid because the degree in \( s \) of the adjoint of \((sI - A)\) is at most \( n - 1 \). Verify that

\[
\begin{align*}
\alpha_1 &= -\frac{\text{tr} AR_0}{1}, & R_0 &= I \\
\alpha_2 &= -\frac{\text{tr} AR_1}{2}, & R_1 &= AR_0 + \alpha_1 I = A + \alpha_1 I \\
\alpha_3 &= -\frac{\text{tr} AR_2}{3}, & R_2 &= AR_1 + \alpha_2 I = A^2 + \alpha_2 A + \alpha_1 I \\
\alpha_{n-1} &= -\frac{\text{tr} AR_{n-2}}{n-1}, & R_{n-1} &= AR_{n-2} + \alpha_{n-2} I = A^{n-1} + \alpha_{n-2} A + \alpha_{n-1} I \\
\alpha_n &= -\frac{\text{tr} AR_{n-1}}{n}, & R_n &= 0 = AR_{n-1} + \alpha_n I 
\end{align*}
\]

where \( \text{tr} \) stands for the trace and is defined as the sum of all the diagonal elements of a matrix. This procedure of computing \( \alpha_i \) and \( R_i \) is called the Leverrier algorithm. \([\text{Hint: The right-hand-side equations can be verified from } \Delta(s)I = (sI - A)(R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2} s + R_{n-1}). \] For a derivation of the left-hand-side equations, see Reference S185.]

2-40 Prove the Cayley–Hamilton theorem. \([\text{Hint: Use Problem 2-39 and eliminate } R_{n-1}, R_{n-2}, \ldots, \text{from } 0 = AR_{n-1} + \alpha_n I] \]

2-41 Show, by using Problem 2-39,

\[(sI - A)^{-1} = \frac{1}{\Delta(s)} [A^{n-1} + (s + \alpha_1)A^{n-2} + (s^2 + \alpha_1 s + \alpha_2)A^{n-3} + \cdots + (s^{n-1} + \alpha_{n-2} s^{n-2} + \cdots + \alpha_n I)] \]

2-42 Let

\[(sI - A)^{-1} = \frac{1}{\Delta(s)} \text{ Adjoint } (sI - A) \]
and let \( m(s) \) be the monic greatest common divisor of all elements of Adjoint \( (sI - A) \). Show that the minimal polynomial of \( A \) is equal to \( \Delta(s) / m(s) \).

2-43 Let all eigenvalues of \( A \) be distinct and let \( q_i \) be a (right) eigenvector of \( A \) associated with \( \lambda_i \), that is, \( Aq_i = \lambda_i q_i \). Define \( Q = [q_1, q_2, \ldots, q_n] \) and define

\[
P = Q^{-1} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}
\]

where \( p_i \) is the \( i \)th row of \( P \). Show that \( p_i \) is a left eigenvector of \( A \) associated with \( \lambda_i \), that is, \( p_i A = \lambda_i p_i \).

2-44 Show that if all eigenvalues of \( A \) are distinct, then \( (sI - A)^{-1} \) can be expressed as

\[
(sI - A)^{-1} = \sum_{s = \lambda_i} \frac{1}{s - \lambda_i} q_i p_i
\]

where \( q_i \) and \( p_i \) are right and left eigenvectors of \( A \) associated with \( \lambda_i \).

2-45 A matrix \( A \) is defined to be cyclic if its characteristic polynomial is equal to its minimal polynomial. Show that \( A \) is cyclic if and only if there is only one Jordan block associated with each distinct eigenvalue.

2-46 Consider the matrix equation

\[
P E P + D P + P F + G = 0
\]

where all matrices are \( n \times n \) constant matrices. It is called an algebraic Riccati equation.

Define

\[
M = \begin{bmatrix} -F & -E \\ G & D \end{bmatrix}
\]

Let

\[
Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}
\]

consist of all generalized eigenvectors of \( M \) so that \( Q^{-1} M Q = J \) is in a Jordan canonical form. We write

\[
\begin{bmatrix} -F & -E \\ G & D \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}
\]

Show that if \( Q_1 \) is nonsingular, then \( P = Q_2 Q_1^{-1} \) is a solution of the Riccati equation.

2-47 Give three different norms of the vector \( x = [1 \quad -4 \quad 3] \).

2-48 Verify the three norms of \( A \) in Figure 2-7.

2-49 Show that the set of all piecewise continuous complex-valued functions defined over \([0, \infty)\) forms a linear space over \( \mathbb{C} \). Show that

\[
\langle g, h \rangle \triangleq \int_0^\infty g^*(t) h(t) \, dt
\]

\textsuperscript{31} See Reference SA.
qualifies as an inner product of the space, where \( g \) and \( h \) are two arbitrary functions of the space. What is the form of the Schwarz inequality in this space?

\[ \textbf{2-50} \] Show that an \( n \times n \) matrix \( A \) has the property \( A^k = 0 \) for \( k > m \) if and only if \( A \) has eigenvalue 0 with multiplicity \( n \) and index \( m \). A matrix with the property \( A^k = 0 \) is called a \textit{nilpotent matrix}. [\textit{Hint: Use Equation (2-64) and Jordan canonical form.}]

\[ \textbf{2-51} \] Let \( A \) be an \( m \times n \) matrix. Show that the set of all \( 1 \times m \) vectors \( y \) satisfying \( yA = 0 \) forms a linear space, called the left null space of \( A \), of dimension \( m - \rho(A) \).