Consider a bicyclist riding at a constant speed up a curved road that climbs into the Black Hills. Over short distances, we can approximate the road’s shape (as viewed from above) by a portion of a circle with radius \( a \) feet. In addition, we will assume that the cyclist gains elevation at the constant rate of \( h \) feet per second. These assumptions lead us to a vector-valued function describing the position of the cyclist at any time \( t \):

\[
\mathbf{r}(t) = a \cos bt \mathbf{i} + a \sin bt \mathbf{j} + ht \mathbf{k}.
\]

We will need a few more assumptions to complete our description of the motion.

a. Specifically, \( \mathbf{r}(t) \) describes the position of the center of the front wheel at time \( t \). The wheel itself will be considered to be 2-dimensional with a wheel radius \( w \) of 13 inches.

b. The constant speed \( v \) of the cyclist is 12 miles per hour.

c. The radius of the curve is \( a = 600 \) feet, and the rate of elevation gain is \( h = 0.5 \) feet per second.

For consistency, convert all times to seconds and all distances to feet.

Let \( P \) be the point on the front tire that is the forward-most point of the bicycle at time \( t = 0 \). Your job is to determine the position and velocity of the point \( P \) as vector-valued functions of time.

The following exercises are designed to lead you through this problem. You should wait until your computations are done to substitute numerical values for \( a, b, \) and \( h \), as well as the parameters \( w, v, \) and \( f \) that are introduced along the way. (Please use exactly these symbols so that I can read your work!) You may work in groups of two or three. Turn in handwritten answers to problems 1-6. For problem 7, e-mail your spreadsheet to donald.teets@sdsmt.edu with math225 (no spaces) in the subject line.

1. (2) Find the value of the constant \( b \).

2. (5) Find the moving reference frame unit vectors \( \mathbf{T}, \mathbf{N} \) and \( \mathbf{B} \). These will be functions of time \( t \). (It’s best to use the definition to find \( \mathbf{N} \) rather than one of the other formulas we developed.) Also, find the derivative \( \mathbf{B}'(t) \) for later use.

3. (2) Determine \( f \), the frequency of rotation of the front tire. (Observe that \( \cos 2\pi ft \) and \( \sin 2\pi ft \) are periodic with frequency \( f \).)

4. (2) Use your answers to questions 2 and 3 to determine the position of the point \( P \) in the \( \mathbf{T} \mathbf{N} \mathbf{B} \) reference frame. Your answer should be a sum of scalar multiples of these vectors. Call it \( \mathbf{P}_{\mathbf{T}\mathbf{N}\mathbf{B}} \). Be careful about the direction of rotation of the point \( P \) in this reference frame.

5. (2) The vector \( \mathbf{P}_{\mathbf{T}\mathbf{N}\mathbf{B}} \) (your answer to question 4) gives the location of the point \( P \) at time \( t \), with respect to the center of the wheel. Use this along with the function \( \mathbf{r}(t) \) to write \( \mathbf{p}(t) \), the position function for the point \( P \) at time \( t \). Express \( \mathbf{p}(t) \) in terms of \( \mathbf{r}, \mathbf{T}, \mathbf{N}, \) and \( \mathbf{B} \).

6. (5) Now that you have \( \mathbf{p}(t) \), use it to find \( \mathbf{v}(t) \), the velocity function for the point \( P \). Express your answer in terms of the vectors \( \mathbf{T}, \mathbf{T}', \mathbf{B}, \) and \( \mathbf{B}' \). (Note: \( \mathbf{r}' = \mathbf{vT} \).)

7. (12) Build a spreadsheet showing numerical values of the vectors \( \mathbf{r}(t), \mathbf{p}(t), \mathbf{T}(t), \) and \( \mathbf{B}(t) \) at times \( t = t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots, t_0 + 30\Delta t \) seconds. Name the parameters \( a, b, h, w, v, \) and \( f \), and use these names in your formulas so I can read them! Put \( t_0 \) and \( \Delta t \) values at the top of the spreadsheet so they can be changed easily. (For starters, just use \( t_0 = 0 \) and \( \Delta t = 0.1 \).) Then put \( t \) values in column A; \( \mathbf{r} \) values in columns B, C, and D; \( \mathbf{T} \) values in columns E, F, and G; \( \mathbf{B} \) values in columns H, I, and J; and \( \mathbf{p} \) values in columns K, L, and M. Put headings on each column to show what they represent.