Math 225 Exam 2A

3/3/11

Name

Key

Question 1 is worth 20 points, and questions 2-6 are 16 points each. Show work as necessary to support answers on questions 2-6.

1. Clearly mark each question True or False.

T F a. The directional derivative is a vector.

T F b. The graph of \( f(x, y) = \sqrt{x^2 + y^2} \) is a cone.

T F c. The level curves of \( f(x, y) = 3x + y - 2 \) are lines in the \( xy \) plane.

T F d. The domain of \( f(x, y) = \sqrt{x+y} \) contains all points inside the unit circle.

T F e. The second derivative test fails if \( D(x, y) = 0 \) at the critical point being tested.

T F f. If \( z = f(x, y), \ x = g(s, t), \) and \( y = h(s, t) \) then \( \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \).

T F g. If \( f(x, y) \) is continuous then it must have an absolute maximum value.

T F h. The gradient vector \( \nabla f \) is always orthogonal to level curves of \( f \).

T F i. A horizontal tangent plane at \( (x_0, y_0) \) requires \( f_x(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) = 0 \).

T F j. The partial derivative is a special case of the directional derivative.

2. For \( f(x, y) = ye^{x^2} + \sin(xy) \), find the mixed partial derivative \( f_{xy} \).

\[
\frac{\partial}{\partial x} = 2xye^{x^2} + y\cos(xy)
\]

\[
\frac{\partial}{\partial y} = 2xe^{x^2} - \left( \sin(xy) + \cos(xy) \right) + \cos(xy)
\]

\[
= 2xe^{x^2} - xy\sin(xy) + \cos(xy)
\]

\[
n = 32
\]

\[
\bar{x} = 67.8
\]
3. Find all critical points for the function \( f(x, y) = x^3 - 2x^2 + y^2 - 2xy \). Classify each critical point as a local maximum, local minimum, or saddle point.

\[
\begin{align*}
 f_x &= 3x^2 - 4x - 2y = 0 \quad \Rightarrow \quad 3x^2 - 6x = 0 \quad 3x(x-2) = 0 \\
 f_y &= 2y - 2x = 0 \Rightarrow y = x \\
 x &= 0 \quad x = 2 \\
 y &= 0 \quad y = 2
\end{align*}
\]

\[
\begin{align*}
 f_{xx} &= 6x - 4 \\
 f_{yy} &= 2 \\
 f_{xy} &= -2 \\
 D(0,0) &= (-4)(2) - (2)^2 = -12 < 0 \quad \text{saddle} \\
 D(2,2) &= 8 \cdot 2 - (-2)^2 = 12 > 0 \quad \text{local max/min} \\
 f_{xx}(2,2) &= 8 \quad \text{loc min}
\end{align*}
\]
4. The temperature at the point \((x, y)\) in a flat plate is given by \(T(x, y) = e^{2x+3y}\).

a. What is the temperature at the origin \((0, 0)\)?

\[ T(0, 0) = 1 \]

b. In what direction from \((0, 0)\) does the temperature increase most rapidly? In what directions from \((0, 0)\) is there no change in temperature?

Most rapidly in the direction of the gradient:
\[ \nabla T = \left< 2e^{2x+3y}, 3e^{2x+3y} \right> \]
so \(\nabla T(0, 0) = \left< 2, 3 \right>\)

No change \(\perp\) to \(\nabla T(0,0):\)
\[ \left< -3, 2 \right> \text{ and } \left< 3, -2 \right> \]

c. Does the temperature increase or decrease as one moves from \((0, 0)\) in the direction of \(\mathbf{v} = \left< -2, 1 \right>\)? What is the rate of increase or decrease?

\[ \hat{u} = \left< -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right> \]

\[ D\nabla T(0,0) = \nabla T(0,0) \cdot \hat{u} \]
\[ = \left< 2, 3 \right> \cdot \left< -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right> \]
\[ = -\frac{1}{\sqrt{5}} \]

decreases \(\frac{1}{\sqrt{5}}\) degrees per unit travelled
(or \(1\) degree per \(\sqrt{5}\) units travelled)
5. Find the absolute maximum and absolute minimum values of \( f(x, y) = x^2 + y^2 - x \) on the domain \( x^2 + y^2 \leq 1 \).

\[
\begin{align*}
\frac{f_x}{x} &= 2x - 1 = 0 & x &= \frac{1}{2} \\
\frac{f_y}{y} &= 2y = 0 & y &= 0
\end{align*}
\]

Boundary:

\( z = 1 - x \quad -1 \leq x \leq 1 \)

\( z' = -1 \)

No critical points. Endpoints: \((-1, 0, 2)\) and \((1, 0, 0)\)

6. The volume of a right circular cylinder is computed using the formula \( V = \pi r^2 h \). The radius and height are measured with possible errors of 4% and 2%, respectively. Use a linear (tangent plane) approximation to estimate the maximum possible percent error in the computed volume.

\[
\frac{\Delta V}{V} = \frac{V_r}{V} \frac{\Delta r}{r} + \frac{V_h}{V} \frac{\Delta h}{h}
\]

\[
= \left( \frac{2\pi rh}{\pi r^2} \right) \frac{\Delta r}{r} + \left( \frac{\pi r^2 h}{\pi r^2} \right) \frac{\Delta h}{h}
\]

\[
= \frac{2 \Delta r}{r} + \frac{\Delta h}{h}
\]

\[
= 2 \times (4\%) + 2\% = 10\%
\]