Exercises for Section 14.4

Exercises 1–4, find the mass of the lamina described by the equations, given that its density is \( \rho(x, y) = xy \). (Hint: Some of the integrals are simpler in polar coordinates.)

1. \( 0 \leq x \leq 4, \ 0 \leq y \leq 3 \)
2. \( x \geq 0, \ 0 \leq y \leq 5 - x^2 \)
3. \( x \geq 0, \ 0 \leq y \leq \sqrt{4 - x^2} \)
4. \( x \geq 0, \ 3 \leq y \leq 3 + \sqrt{9 - x^2} \)

Exercises 5–8, find the mass and center of mass of the lamina:
- Each density.
- \( R \): rectangle with vertices \((0, 0), (a, 0), (0, b), (a, b)\)
  (a) \( \rho = k \) (b) \( \rho = ky \) (c) \( \rho = kx \)
- \( R \): rectangle with vertices \((0, 0), (a, 0), (0, b), (a, b)\)
  (a) \( \rho = kxy \) (b) \( \rho = k(x^2 + y^2) \)
- \( R \): triangle with vertices \((0, 0), (b/2, h), (b, 0)\)
  (a) \( \rho = k \) (b) \( \rho = ky \) (c) \( \rho = kx \)
- \( R \): triangle with vertices \((0, 0), (0, a), (a, 0)\)
  (a) \( \rho = k \) (b) \( \rho = x^2 + y^2 \)

Translations in the Plane: Translate the lamina in Exercise 5 to the right five units and determine the resulting center of mass.

1. Conjecture: Use the result of Exercise 9 to make a conjecture about the change in the center of mass when a lamina of constant density is translated \( h \) units horizontally or \( k \) units vertically. Is the conjecture true if the density is not constant? Explain.

Exercises 11–22, find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density or densities. (Hint: Some of the integrals are simpler in polar coordinates.)

1. \( y = \sqrt{a^2 - x^2}, \ y = 0 \)
   (a) \( \rho = k \) (b) \( \rho = k(a - y) \)
2. \( x^2 + y^2 = a^2, \ 0 \leq x, \ 0 \leq y \)
   (a) \( \rho = k \) (b) \( \rho = k(x^2 + y^2) \)
3. \( y = \sqrt{x}, \ y = 0, \ x = 4, \ \rho = kxy \)
4. \( y = x^2, \ y = 0, \ x = 2, \ \rho = kx \)
5. \( y = \frac{1}{1 + x^2}, \ y = 0, \ x = -1, \ x = 1, \ \rho = k \)
6. \( xy = 4, \ x = 1, \ x = 4, \ \rho = kx^2 \)
7. \( x = 16 - y^2, \ x = 0, \ \rho = kx \)
8. \( y = 9 - x^2, \ y = 0, \ \rho = ky^2 \)
9. \( y = \sin \frac{\pi}{L}, \ y = 0, \ x = 0, \ x = L, \ \rho = ky \)
10. \( y = \cos \frac{\pi}{L}, \ y = 0, \ x = 0, \ x = \frac{L}{2}, \ \rho = k \)
11. \( y = \sqrt{a^2 - x^2}, \ 0 \leq y \leq x, \ \rho = k \)
12. \( y = \sqrt{a^2 - x^2}, \ y = 0, \ x = a, \ \rho = kx \)
13. \( y = e^{-x}, \ y = 0, \ x = 0, \ x = 2, \ \rho = ky \)
14. \( y = \ln x, \ y = 0, \ x = 1, \ \rho = kx \)
15. \( r = 2 \cos \theta, \ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}, \ \rho = k \)
16. \( r = 1 + \cos \theta, \ \rho = k \)

In Exercises 23–26, use a computer algebra system to find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density.

23. \( y = e^{-x}, \ y = 0, \ x = 0, \ x = 2, \ \rho = ky \)
24. \( y = \ln x, \ y = 0, \ x = 1, \ \rho = kx \)
25. \( r = 2 \cos \theta, \ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}, \ \rho = k \)
26. \( r = 1 + \cos \theta, \ \rho = k \)

In Exercises 27–32, verify the given moment(s) of inertia and find \( \bar{x} \) and \( \bar{y} \). Assume that each lamina has a density of \( \rho = 1 \). (These regions are common shapes used in engineering.)

27. Rectangle
   \[ I_x = \frac{1}{12} bh^3, \quad I_y = \frac{1}{12} b^3 h \]

28. Right triangle
   \[ I_x = \frac{1}{12} bh^3, \quad I_y = \frac{1}{12} b^3 h \]

29. Circle
   \[ I_x = \frac{1}{2} \pi a^4, \quad I_y = \frac{1}{2} \pi a^4 \]

30. Semicircle
   \[ I_x = \frac{1}{2} \pi a^4, \quad I_y = \frac{1}{2} \pi a^4 \]

31. Quarter circle
   \[ I_x = \frac{1}{2} \pi a^4, \quad I_y = \frac{1}{2} \pi a^4 \]

32. Ellipse
   \[ I_x = \frac{1}{2} \pi ab(a^2 + b^2) \]

In Exercises 33–40, find \( I_x, I_y, I_{xy}, \bar{x}, \bar{y} \) for the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integrals.

33. \( y = 0, \ y = b, \ x = 0, \ x = a, \ \rho = ky \)
34. \( y = \sqrt{a^2 - x^2}, \ y = 0, \ \rho = ky \)
35. \( y = 4 - x^2, \ y = 0, \ x > 0, \ \rho = kx \)
36. \( y = x, \ y = x^2, \ \rho = kxy \)
37. \( y = \sqrt{x}, \ y = 0, \ x = 4, \ \rho = kxy \)
38. \( y = x^2, \ y^2 = x, \ \rho = kx \)
39. \( y = x^2, \ y^2 = x, \ \rho = kx \)
40. \( y = x^3, \ y = 4x, \ \rho = k|y| \)
in Exercises 31 and 32, the figure shows the region of integration for the given integral. Rewrite the integral as an equivalent iterated integral in the five other orders.

31. \[ \int_0^1 \int_0^1 \int_0^{1-y} dz
dx
dy \]
32. \[ \int_0^2 \int_0^1 \int_0^{1-y} dz
dx
dy \]

\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]
\[ z = 1 - y \]
\[ x = 1 - y^2 \]

\[ \text{Mass and Center of Mass} \quad \text{In Exercises 33–36, find the mass and the indicated coordinates of the center of mass of the solid of given density bounded by the graphs of the equations.} \]

33. Find \( \bar{x} \) using \( \rho(x, y, z) = k \).
   \[ \frac{2x + 3y + 6z}{12} \]
   \[ x = 0, y = 0, z = 0 \]

34. Find \( \bar{y} \) using \( \rho(x, y, z) = ky \).
   \[ \frac{3x + 3y + 5z}{15} \]
   \[ x = 0, y = 0, z = 0 \]

35. Find \( \bar{z} \) using \( \rho(x, y, z) = kx \).
   \[ \frac{z = 4 - x, z = 0, y = 0, y = 4, x = 0} {b} \]

36. Find \( \bar{y} \) using \( \rho(x, y, z) = k \).
   \[ \frac{x + y + z}{a} \]
   \[ (a, b, c > 0), x = 0, y = 0, z = 0 \]

\[ \text{Mass and Center of Mass} \quad \text{In Exercises 37 and 38, set up the triple integrals for finding the mass and the center of mass of the solid bounded by the graphs of the equations.} \]

37. \( x = 0, x = b, y = 0, y = b, z = 0, z = b \)
   \[ \rho(x, y, z) = kxy \]
38. \( x = 0, x = a, y = 0, y = b, z = 0, z = c \)
   \[ \rho(x, y, z) = kcz \]

\[ \text{Think About It: The center of mass of a solid of constant density is shown in the figure. In Exercises 39–42, make a conjecture about how the center of mass (\( \bar{x}, \bar{y}, \bar{z} \)) will change for the nonconstant density \( \rho(x, y, z) \). Explain.} \]

39. \( \rho(x, y, z) = kx \)
40. \( \rho(x, y, z) = ky \)
41. \( \rho(x, y, z) = k(y + 2) \)
42. \( \rho(x, y, z) = kxy^2(y + 2)^2 \)

\[ (2, 0, \frac{5}{3}) \]

\[ \text{Centroid} \quad \text{In Exercises 43–48, find the centroid of the solid region bounded by the graphs of the equations or described by the figure. Use a computer algebra system to evaluate the triple integrals. (Assume uniform density and find the center of mass.)} \]

43. \( z = \frac{h}{3} \sqrt{x^2 + y^2}, z = h \)
44. \( y = \sqrt{4 - x^2}, z = y, z = 0 \)
45. \( z = \sqrt{4 - x^2 - y^2}, z = 0 \)
46. \( z = \frac{1}{y^2 + 1}, z = 0, x = -2, x = 2, y = 0, y = 1 \)
47. \[ 20 \text{cm} \]
48. \[ 12 \text{cm} \]

\[ (0, 4, 0) \]

\[ (0, 0, 0) \]

\[ (0, 3, 0) \]

\[ (5, 0, 0) \]

49. (a) \( \rho = k \)
   (b) \( \rho = kxyz \)

50. (a) \( \rho(x, y, z) = k \)
   (b) \( \rho(x, y, z) = k(x^2 + y^2) \)

51. (a) \( \rho(x, y, z) = k \)
   (b) \( \rho = ky \)

52. (a) \( \rho = k \)
   (b) \( \rho = k(4 - z) \)

\[ \text{Moments of Inertia} \quad \text{In Exercises 53 and 54, verify the moments of inertia for the solid of uniform density. Use a computer algebra system to evaluate the triple integrals.} \]

53. \( I_x = \frac{1}{12} \pi (3a^2 + L^2) \)
54. \( I_y = \frac{1}{12} \pi (3a^2 + L^2) \)

\[ \text{Moments of Inertia} \quad \text{In Exercises 53 and 54, verify the moments of inertia for the solid of uniform density. Use a computer algebra system to evaluate the triple integrals.} \]

53. \( I_x = \frac{1}{12} \pi (3a^2 + L^2) \)
54. \( I_y = \frac{1}{12} \pi (3a^2 + L^2) \)
54. \[ I_x = \frac{1}{12}ma^2 + b^2 \]
\[ I_y = \frac{1}{12}m(b^2 + c^2) \]
\[ I_z = \frac{1}{12}ma^2 + c^2 \]

**Moments of Inertia** In Exercises 55 and 56, set up a triple integral that gives the moment of inertia about the z-axis of the solid region \( Q \) of density \( \rho \).

55. \( Q = \{(x, y, z) : -1 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq z \leq 1 - x \} \)
\[ \rho = \sqrt{x^2 + y^2 + z^2} \]
56. \( Q = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 4 - x^2 - y^2 \} \)
\[ \rho = kz^2 \]

In Exercises 57 and 58, using the description of the solid region, set up the integral for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

57. The solid bounded by \( z = 4 - x^2 - y^2 \) and \( z = 0 \) with density function \( \rho = k \).
58. The solid in the first octant bounded by the coordinate planes and \( x^2 + y^2 + z^2 = 25 \) with density function \( \rho = kxy \).

**Writing About Concepts**

59. Define a triple integral and describe a method of evaluating a triple integral.
60. Give the number of possible orders of integration when evaluating a triple integral.
61. Consider solid \( A \) and solid \( B \) of equal weight shown below.
   (a) Because the solids have the same weight, which has the greater density?
   (b) Which solid has the greater moment of inertia? Explain.
   (c) The solids are rolled down an inclined plane. They are started at the same time and at the same height. Which will reach the bottom first? Explain.

**Writing About Concepts (continued)**

62. Determine whether the moment of inertia about the z-axis of the cylinder in Exercise 53 will increase or decrease the nonconstant density \( \rho(x, y, z) = \sqrt{x^2 + z^2} \) and \( a \).

**Average Value** In Exercises 63--66, find the average value function over the given solid. The average value of a continuous function \( f(x, y, z) \) over a solid region \( Q \) is
\[ \frac{1}{V} \iiint_Q f(x, y, z) \, dV \]
where \( V \) is the volume of the solid region \( Q \).

63. \( f(x, y, z) = z^2 + 4 \) over the cube in the first octant bounded by the coordinate planes, and the planes \( x = 1, y = 1, \) and \( z = 1 \).
64. \( f(x, y, z) = xyz \) over the cube in the first octant bounded by the coordinate planes, and the planes \( x = 3, y = 3, \) and \( z = 3 \).
65. \( f(x, y, z) = x + y + z \) over the tetrahedron in the first octant with vertices \((0, 0, 0), (2, 0, 0), (0, 2, 0),\) and \((0, 0, 2)\).
66. \( f(x, y, z) = x + y \) over the solid bounded by the planes \( x^2 + y^2 + z^2 = 2 \).

67. Find the solid region \( Q \) where the triple integral
\[ \iiint_Q (1 - 2x^2 - y^2 - 3z^2) \, dV \]
is a maximum. Use a computer algebra system to apply the maximum value. What is the exact maximum value?

68. Find the solid region \( Q \) where the triple integral
\[ \iiint_Q (1 - x^2 - y^2 - z^2) \, dV \]
is a maximum. Use a computer algebra system to apply the maximum value. What is the exact maximum value?

69. Solve for \( a \) in the triple integral
\[ \int_0^1 \int_0^1 \int_a^1 dz \, dx \, dy = \frac{14}{15} \]
70. Determine the value of \( b \) so that the volume of the ellipsoid
\[ x^2 + \frac{y^2}{b^2} + \frac{z^2}{9} = 1 \]
is \( 16\pi \).

**Putnam Exam Challenge**

71. Evaluate
\[ \lim_{n \to \infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos \left[ \frac{\pi}{2n} (x_1 + x_2 + \cdots + x_n) \right] \, dx_1 \, dx_2 \cdots \, dx_n \]

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Exercises for Section 14.7

In Exercises 1–6, evaluate the iterated integral.

1. \[
\int_0^\pi \int_0^{\pi/2} r \cos \theta \, dr \, d\theta
\]
2. \[
\int_0^\pi \int_0^{\pi/2} r z \, dz \, d\theta
\]
3. \[
\int_0^{\pi/2} \int_0^\pi r \sin \theta \, dz \, d\theta
\]
4. \[
\int_0^\pi \int_0^\pi e^{r^2} \rho^2 \, d\phi \, d\rho
\]
5. \[
\int_0^\pi \int_0^\pi \rho^2 \sin \phi \, d\phi \, d\theta
\]
6. \[
\int_0^\pi \int_0^\pi \rho^2 \sin \phi \cos \phi \, d\phi \, d\theta
\]

In Exercises 7 and 8, use a computer algebra system to evaluate the iterated integral.

7. \[
\int_0^{\pi/2} \int_0^\pi r e^r \, d\theta \, dr
\]
8. \[
\int_0^{\pi/2} \int_0^\pi (2 \cos \phi) \rho^2 \, d\phi \, d\rho
\]

In Exercises 9–12, sketch the solid region whose volume is given by the iterated integral, and evaluate the iterated integral.

9. \[
\int_0^{\pi/2} \int_0^\pi \int_0^r r \, dz \, d\theta \, dr
\]
10. \[
\int_0^{2\pi} \int_0^\pi \int_0^r r \, dz \, d\theta \, dr
\]
11. \[
\int_0^{2\pi} \int_0^{\pi/4} \int_0^\rho \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta
\]
12. \[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^\rho \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta
\]

In Exercises 13–16, convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

13. \[
\int_0^\pi \int_0^\pi \int_{-\sqrt{\rho^2 - r^2}}^{\sqrt{\rho^2 - r^2}} x \, dx \, dy \, dz
\]
14. \[
\int_0^\pi \int_0^\pi \int_{-\sqrt{\rho^2 - r^2}}^{\sqrt{\rho^2 - r^2}} \sqrt{x^2 + y^2} \, dx \, dy \, dz
\]
15. \[
\int_0^\pi \int_0^\pi \int_{-\sqrt{\rho^2 - r^2}}^{\sqrt{\rho^2 - r^2}} x \, dz \, dy \, dx
\]
16. \[
\int_0^\pi \int_0^\pi \int_{-\sqrt{\rho^2 - r^2}}^{\sqrt{\rho^2 - r^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx
\]

Volume In Exercises 17–20, use cylindrical coordinates to find the volume of the solid.

17. Solid inside both \( x^2 + y^2 + z^2 = a^2 \) and \((x - a/2)^2 + y^2 = (a/2)^2\)
18. Solid inside \( x^2 + y^2 + z^2 = 16 \) and outside \( z = \sqrt{x^2 + y^2} \)
19. Solid bounded by the graphs of the sphere \( r^2 + z^2 = a^2 \) and the cylinder \( r = a \cos \theta \)
20. Solid inside the sphere \( x^2 + y^2 + z^2 = 4 \) and above the upper nappe of the cone \( z^2 = x^2 + y^2 \)

Mass In Exercises 21 and 22, use cylindrical coordinates to find the mass of the solid \( Q \).

21. \( Q = \{(x, y, z): 0 \leq z \leq 2x - 2y, x^2 + y^2 \leq 4\} \)
\[\rho(x, y, z) = k \sqrt{x^2 + y^2}\]
22. \( Q = \{(x, y, z): 0 \leq z \leq 12e^{-6}, x^2 + y^2 \leq 4, x \geq 0, y \geq 0\} \)
\[\rho(x, y, z) = k\]

In Exercises 23–28, use cylindrical coordinates to find the indicated characteristic of the cone shown in the figure.

23. Volume Find the volume of the cone.
24. Centroid Find the centroid of the cone.
25. Center of Mass Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the axis of the cone. Use a computer algebra system to evaluate the triple integral.
26. Center of Mass Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the base. Use a computer algebra system to evaluate the triple integral.
27. Moment of Inertia Assume that the cone has uniform density and show that the moment of inertia about the \( z \)-axis is \( I_z = \frac{3}{10} m r_0^2 \).
28. Moment of Inertia Assume that the density of the cone is \( \rho(x, y, z) = k \sqrt{x^2 + y^2} \) and find the moment of inertia about the \( z \)-axis.

Moment of Inertia In Exercises 37 and 38, use spherical coordinates to find the moment of inertia about the \( z \)-axis of the lid of uniform density.

37. Solid bounded by the hemisphere \( \rho = \cos \phi, \pi/4 \leq \phi \leq \pi/2, \) and the cone \( \phi = \pi/6 \)
38. Solid lying between two concentric hemispheres of radii \( r \) and \( R \), where \( r < R \).