1. Calculate the following limits.

(a) \[
\lim_{x \to 0} \frac{3\sin x}{2x} = \frac{3}{2} \cdot \lim_{x \to 0} \frac{\sin x}{x} = \frac{3}{2} \cdot 1 = \frac{3}{2},
\]

(b) \[
\lim_{x \to 0} \sin \left(\frac{1}{x}\right)
\]

Let \( y = \frac{1}{x} \). Then as \( x \to 0, y \to \infty \). Since \[
\lim_{y \to \infty} \sin y
\]
does not exist, then \[
\lim_{x \to 0} \sin \left(\frac{1}{x}\right)
\]
does not exist. Or, consider the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin \left(\frac{1}{x}\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.5440</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.5064</td>
</tr>
<tr>
<td>-0.001</td>
<td>-0.8269</td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.3056</td>
</tr>
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</tr>
</tbody>
</table>

Clearly, there is no pattern here, and the limit does not exist.

(c) \[
\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3x - 1} - \sqrt{2x + 1}} = \lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3x - 1} - \sqrt{2x + 1}} \cdot \frac{\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1}}{\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1}}
\]

\[
= \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1})}{(x^2 + 3x - 1) - (2x + 1)}
\]

\[
= \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1})}{x^2 + x - 2}
\]

\[
= \lim_{x \to 1} \frac{(x - 1)(\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1})}{(x + 2)(x - 1)}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2 + 3x - 1} + \sqrt{2x + 1}}{x + 2}
\]

\[
= \frac{\sqrt{3} + \sqrt{3}}{3}
\]

\[
= \frac{2\sqrt{3}}{3}.
\]
(d)

\[
\lim_{x \to 0} \frac{1 + \sin x - \cos x}{\sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \cdot \frac{\sin x}{1 + \cos x} + 1 = \frac{1 - \cos^2 x}{\sin x} + 1
\]

\[
= \lim_{x \to 0} \sin^2 x + 1
\]

\[
= \lim_{x \to 0} \sin x + 1
\]

\[
= 0 + 1 + 1 = 1.
\]

(e)

\[
\lim_{x \to 0} \frac{x^4 - x^2 - 12}{x - 2} = \lim_{x \to 2} (x - 2)(x^3 + 2x^2 + 3x + 6) = \lim_{x \to 2} (x^3 + 2x^2 + 3x + 6) = 28.
\]

Use long division with this problem.

(f)

\[
\lim_{h \to 0} \frac{\sqrt{16 - h} - \sqrt{16}}{h} = \lim_{h \to 0} \frac{\sqrt{16 - h} - \sqrt{16}}{h} \cdot \frac{\sqrt{16 - h} + \sqrt{16}}{\sqrt{16 - h} + \sqrt{16}}
\]

\[
= \lim_{h \to 0} \frac{(16 - h) - 16}{h(\sqrt{16 - h} + \sqrt{16})}
\]

\[
= \lim_{h \to 0} \frac{-h}{h(\sqrt{16 - h} + \sqrt{16})}
\]

\[
= \lim_{h \to 0} \frac{-1}{\sqrt{16 - h} + \sqrt{16}}
\]

\[
= -\frac{1}{4 + 4} = -\frac{1}{8}.
\]

(g)

\[
\lim_{t \to 3^+} \frac{1}{\sqrt{t^2 - t - 6}} = \lim_{t \to 3^+} \frac{1}{\sqrt{(t-3)(t+2)}}
\]

As \(t \to 3^+\), then \(t - 3\) becomes a very small positive number. Thus the denominator becomes a very small number, and the overall value becomes increasingly large. Therefore

\[
\lim_{t \to 3^+} \frac{1}{\sqrt{t^2 - t - 6}} = \infty.
\]

(h)

\[
\lim_{x \to \pi/4} x \cot x = \frac{\pi \cos (\pi/4)}{4 \sin (\pi/4)} = \frac{\pi \sqrt{2}/2}{4 \sqrt{2}/2} = \frac{\pi}{4}.
\]
\[
\lim_{x \to 0} x \cot (17x) = \lim_{x \to 0} x \frac{\cos (17x)}{\sin (17x)} = \lim_{x \to 0} x \frac{\cos (17x)}{\sin (17x)} \cdot \frac{17}{17} = \lim_{x \to 0} \frac{\cos (17x)}{17} \cdot \frac{17}{17} = \lim_{x \to 0} \frac{\cos (17x)}{17} \cdot \lim_{x \to 0} \frac{\sin (17x)}{17x} = \left( \lim_{x \to 0} \frac{\cos (17x)}{17} \right) \left( \lim_{x \to 0} \frac{\sin (17x)}{17x} \right) = \frac{1}{17} \cdot 1 = \frac{1}{17}.
\]

\[
\lim_{x \to \infty} \frac{17x^3 + 4x}{8 - 2x - 4x^3} = \lim_{x \to \infty} \frac{x^3(17 + 4/x^2)}{x^3(8/x^3 - 2/x^2 - 4)} = \frac{17 + 0}{0 - 0 - 4} = \frac{-17}{4}.
\]

\[
\lim_{x \to \infty} \frac{6x^2 - 3x + 7}{\sqrt{9x^4 - x^2}} = \lim_{x \to \infty} \frac{x^2(6 - 3/x + 7/x^2)}{\sqrt{x^4(9 - 1/x^2)}} = \lim_{x \to \infty} \frac{x^2(6 - 3/x + 7/x^2)}{x^2 \sqrt{9 - 1/x^2}} = \lim_{x \to \infty} \frac{6 - 3/x + 7/x^2}{\sqrt{9 - 1/x^2}} = \frac{6 - 0 + 0}{\sqrt{9 - 0}} = \frac{6}{3} = 2.
\]

\[
\lim_{x \to \infty} \frac{17x^5 - 12x^3 + 17x^2}{18 + 22x^4 - 4x^6} = \lim_{x \to \infty} \frac{x^5(17 - 12/x^2 + 17/x^3)}{x^6(18/x^6 + 22/x^2 - 4)} = \frac{17 - 12/x^2 + 17/x^3}{x(18/x^6 + 22/x^2 - 4)} = \lim_{x \to \infty} \frac{17 - 0 + 0}{x(0 + 0 - 4)} = \lim_{x \to \infty} \frac{17}{-4x} = 0.
\]

2. Determine the points at which the following functions are not continuous and state the type of discontinuity.

(a) \[f(x) = \frac{x^2 - 1}{x - 3}\]

\[
\lim_{x \to 3^-} f(x) = -\infty \quad \lim_{x \to 3^+} f(x) = \infty,
\]

and so \(f(x)\) is discontinuous at \(x = 3\), which is an infinite discontinuity.
(b) \[ g(x) = \frac{x^2 - x - 6}{x + 2} = \frac{(x + 2)(x - 3)}{x + 2}. \]

Note that \[ \lim_{x \to -2} g(x) = \lim_{x \to -2} \frac{1}{x - 3} = -\frac{1}{5}, \]

but \( g(-2) \) is undefined. Thus \( x = -2 \) is a removable discontinuity.

(c) \[ h(x) = \frac{1 - x}{x^2 + 4x - 5} = \frac{-(x - 1)}{(x + 5)(x - 1)}, \]

and so \( x = 1 \) is a removable discontinuity, but \( x = -5 \) is an infinite discontinuity because \[ \lim_{x \to -5^-} h(x) = \infty \quad \lim_{x \to -5^+} h(x) = -\infty. \]

3. \[ g(x) = \begin{cases}  \ -x^2 + 1 & \text{if } x < 0 \\  \ 1 & \text{if } x = 0 \\  \ \sqrt{x} & \text{if } 0 < x < 4 \\  \ x - 2 & \text{if } x > 4 \end{cases} \]

(a) Is \( g(x) \) continuous at \( x = 0 \)? Why?

\[ \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (-x^2 + 1) = 1 \quad \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \sqrt{x} = 0, \]

and because the limits do not agree, \( g(x) \) is not continuous at \( x = 0 \).

(b) Is \( g(x) \) continuous at \( x = 4 \)? Why?

\[ \lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} \sqrt{x} = 2 \quad \lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x - 2) = 2, \]

and because \( g(4) = \sqrt{4} = 2 \), then \( \lim_{x \to 4} g(x) = g(4) \) and \( g(x) \) is continuous at \( x = 4 \).

(c) Sketch a graph of \( g(x) \). Be sure to include either a readable scale or label appropriate points.

4. Consider the function

\[ g(x) = \begin{cases}  \ x^2 + 8 & \text{if } x < 1 \\  \ 10 - x & \text{if } 1 \leq x < 2 \\  \ 7 & \text{if } x = 2 \\  \ 6x - x^2 & \text{if } x > 2. \end{cases} \]

Is \( g(x) \) continuous at \( x = 2 \)? Why or why not?

\[ \lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (10 - x) = 8 \quad \lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (6x - x^2) = 8, \]

and \( g(2) = 7 \). Thus the left hand and right hand limits agree, but they are not equal to function at \( x = 2 \), thus the function is discontinuous. This is a removable/point discontinuity.

5. Use the definition of the derivative to compute \( f'(x) \) if \( f(x) = x^3 - 6x^2 + 3 \). Use your answer to find the equation of the tangent line to \( f(x) \) at \( x = 2 \).
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{[(x + h)^3 - 6(x + h)^2 + 3] - [x^3 - 6x^2 + 3]}{h} \]
\[ = \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - 6(x^2 + 2xh + h^2) + 3] - [x^3 - 6x^2 + 3]}{h} \]
\[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 6x^2 - 12xh - 6h^2 + 3 - x^3 + 6x^2 - 3}{h} \]
\[ = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 12xh - 6h^2}{h} \]
\[ = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 12x - 6h)}{h} \]
\[ = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 12x + 6h) \]
\[ = 3x^2 - 12x. \]

Thus \( f'(2) = 3(2)^2 - 12(2) = 12 - 24 = -12, \) and thus \( y - f(2) = -12(x - 2) \) becomes \( y = -12x + 11. \)

6. Find the equation of the tangent line of \( f(x) = x^2 + 1 \) at \( x = 4. \)

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{(x + h)^2 + 1 - [x^2 + 1]}{h} \]
\[ = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \]
\[ = \lim_{h \to 0} \frac{h(2x + h)}{h} \]
\[ = \lim_{h \to 0} (2x + h) = 2x. \]

Thus the slope of the tangent line at \( x = 4 \) is \( f'(4) = 8. \) Since the point on the curve is \((4, 17),\) then the equation of the tangent line is \( y - 17 = 8(x - 4), \) which simplifies to \( y = 8x - 15. \)

7. Use the definition of the derivative to compute \( f'(x) \) if \( f(x) = x^{1/3}. \)

Hint: Cube roots have their own form of the conjugate. The conjugate of \( \sqrt[3]{a} - \sqrt[3]{b} \)
is \( a^{2/3} + a^{1/3} \cdot b^{1/3} + b^{2/3}. \)

For example, the conjugate of \( \sqrt[3]{x^2} - \sqrt[3]{3} \)
is \( (x^2)^{2/3} + (x^2)^{1/3} \cdot (3)^{1/3} + (3)^{2/3} = x^{4/3} + x^{2/3} \cdot \sqrt[3]{3} + \sqrt[3]{9}, \)
and so \( (\sqrt[3]{x^2} - \sqrt[3]{3}) \cdot (x^{4/3} + x^{2/3} \cdot \sqrt[3]{3} + \sqrt[3]{9}) = x^2 - 3. \)
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h} \]
\[ = \lim_{h \to 0} \frac{(x + h)^{1/3} - x^{1/3}}{h} \frac{\frac{2}{3}(x + h)^{2/3} + x^{2/3}}{(x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3}} \]
\[ = \lim_{h \to 0} \frac{(x + h) + (x + h)^{2/3}x^{1/3} + (x + h)^{1/3}x^{2/3} - x^{1/3}(x + h)^{2/3} - (x + h)^{1/3}x^{2/3} - x}{h((x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3})} \]
\[ = \lim_{h \to 0} \frac{h}{h((x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3})} \]
\[ = \lim_{h \to 0} \frac{1}{(x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3}} \]
\[ = \frac{1}{x^{2/3} + x^{1/3}x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}. \]

Thus
\[ f'(x) = \frac{1}{3}x^{-2/3}. \]

8. Let
\[ s(x) = \frac{x^2 + 10x + 21}{x - 5}. \]

(a) State why limit laws fail to produce an answer for
\[ \lim_{s \to 5} s(x). \]
Be specific without going overboard.
The law that allows us to consider the numerator and denominator separately fails because the limit of the denominator will be zero.

(b) How would you find the limit given in (a)? What method would you use? Find the limit.
We could either reason out the answer or use a table. Either way,
\[ \lim_{x \to 5^-} s(x) = -\infty \quad \lim_{x \to 5^+} s(x) = \infty, \]
and thus the limit does not exist.