1. Prove the following orthogonality conditions.

   (a) \[ \int_0^L \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m 
eq 0 \\ \frac{L}{2} & n = m = 0 \end{cases} \]

   (b) For \( n = m \neq 0 \) and \( n \neq m \),

   \[ \int_0^L \cos \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx = 0. \]

2. [2.3.1] Derive the ordinary differential equations implied by the use of separation of variables for the following partial differential equations.

   (a) \[ \frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^4} \]

   (b) \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x} \]

3. [2.3.3(d)] Consider the heat equation

   \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \]

subject to the boundary conditions

   \[ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0. \]

Solve the initial value problem if the temperature is initially

   \[ u(x, 0) = \begin{cases} 1 & 0 < x < L/2 \\ 2 & L/2 < x < L. \end{cases} \]

4. Solve for \( u_2(x, y) \) in the example from Section 2.5.1.

5. [2.5.3] Solve Laplace’s equation outside a circular disk \((r \geq a)\) subject to the boundary condition \( u(a, \theta) = f(\theta) \). Assume \( u(r, \theta) \) remains finite as \( r \to \infty \).

6. Solve Laplace’s equation inside a circular annulus \((a < r < b)\) subject to the boundary conditions

   \[ u(a, \theta) = f(\theta) \quad \text{and} \quad u(b, \theta) = 0. \]

7. [2.3.8] Consider

   \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u. \]

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° or with insulated sides with a heat sink proportional to the temperature. Suppose the boundary conditions are

   \[ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0. \]

Suppose the initial condition is \( u(x, 0) = f(x) \). Assuming \( \alpha > 0 \), solve the partial differential equation.