1. Consider a fluid of density $\rho(x, y, z, t)$ and a three-dimensional region of interest, denoted by $\mathcal{V}$. If in addition there is a velocity field given by $\mathbf{v}(x, y, z, t)$, derive a PDE that describes the conservation of mass for the fluid, known as the Continuity Equation for a fluid.

2. Solve

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

with

$$u(0, t) - \frac{\partial u}{\partial x}(0, t) = 0$$
$$u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0$$

and $u(x, 0) = 100$. Determine the first five terms of the series solution and plot (in both $x$ and $t$) in an appropriate program, such as Maple. For plotting purposes, let $k = 0.001$. Don’t expect your approximation to be much of a solution, though, since we only have five terms.

3. [7.3.2(b)] Consider the heat equation in a three-dimensional box-shaped region given by $0 < x < L$, $0 < y < H$, and $0 < z < W$,

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) ,$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z)$$

and subject to the insulated boundary conditions:

$$\frac{\partial u}{\partial x}(0, y, z, t) = 0 \quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0 \quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0$$
$$\frac{\partial u}{\partial x}(L, y, z, t) = 0 \quad \frac{\partial u}{\partial y}(x, H, z, t) = 0 \quad \frac{\partial u}{\partial z}(x, y, W, t) = 0$$

Caution: you will need to check all cases for the eigenvalues of some of the ODE’s.

4. [7.7.5] Solve the vibrating membrane problem on a sector of an annulus given by $a < r < b$ and $0 < \theta < \pi/2$ subject to the boundary condition that $u(r, \theta, t) = 0$ on the entire boundary. (There are no initial conditions, so you don’t need to solve for any coefficients in the solution.)

5. [7.9.2(c)] Solve Laplace’s equation inside a semicircular cylinder $(0 < r < a, 0 < \theta < \pi, 0 < z < H)$ subject to the boundary conditions:

$$\frac{\partial u}{\partial r}(r, \theta, 0) = 0 \quad \frac{\partial u}{\partial r}(r, 0, z) = 0 \quad \frac{\partial u}{\partial r}(a, \theta, z) = \beta(\theta, z)$$
$$\frac{\partial u}{\partial z}(r, \theta, H) = 0 \quad \frac{\partial u}{\partial z}(r, \pi, z) = 0$$

6. Find a series solution to the ODE

$$y'' - 3y' = 0$$

using a Taylor Series expansion and the method outlined in class for finding a formula for $J_m(z)$. Since $x = 0$ is not a singular point for this ODE, you do not need to use the Method of Frobenius. Instead assume $y = \sum_{n=0}^{\infty} a_n x^n$. But be careful, since some of the terms in your derivatives will be zero, this will alter the starting index of your series.
7. A general second-order PDE can be written in the form 

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = G, \]

where \(A, \ldots, G\) are functions of \(x\) and \(y\), though they could be constants. Second-order PDE’s are classified into three types based on the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>Canonical form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic</td>
<td>(B^2 - 4AC &gt; 0)</td>
</tr>
<tr>
<td>Parabolic</td>
<td>(B^2 - 4AC = 0)</td>
</tr>
<tr>
<td>Elliptic</td>
<td>(B^2 - 4AC &lt; 0)</td>
</tr>
</tbody>
</table>

Consider the following examples.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Name</th>
<th>(B^2 - 4AC)</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_{xx} - u_t = 0)</td>
<td>Heat</td>
<td>(0^2 - 4(1)(0) = 0)</td>
<td>parabolic</td>
</tr>
<tr>
<td>(u_{xx} - u_{tt} = 0)</td>
<td>Wave</td>
<td>(0^2 - 4(1)(-1) &gt; 0)</td>
<td>hyperbolic</td>
</tr>
<tr>
<td>(u_{xx} + u_{yy} = 0)</td>
<td>Laplace</td>
<td>(0^2 - 4(1)(1) &lt; 0)</td>
<td>elliptic</td>
</tr>
<tr>
<td>(xu_{xx} + u_{yy} = 0)</td>
<td></td>
<td>(0^2 - 4(x)(1) = -4x)</td>
<td>elliptic for (x &gt; 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>parabolic for (x = 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>hyperbolic for (x &lt; 0)</td>
</tr>
</tbody>
</table>

Classify the following PDE’s.

(a) \(u_{xx} - u_{xy} = 0\)
(b) \(u_{tt} = u_{xx} + u_x + u\)
(c) \(u_{xx} + 3u_{yy} = \sin x\)
(d) \(u_{xx} - 2u_{xy} + u_{yy} = f(x, y)\)
(e) \(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r, \theta)\)

8. Each type of PDE also comes with a canonical form. Many methods have been developed to solve PDE’s in canonical form, and some computer algebra systems require you to enter a PDE in canonical form before the software can solve the PDE. Note that the hyperbolic type has two different canonical forms.

<table>
<thead>
<tr>
<th>Type</th>
<th>Canonical form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic</td>
<td>(u_{aa} - u_{bb} = \Phi(a, b, u, u_a, u_b))</td>
</tr>
<tr>
<td></td>
<td>(u_{ab} = \Phi(a, b, u, u_a, u_b))</td>
</tr>
<tr>
<td>Parabolic</td>
<td>(u_{bb} = \Phi(a, b, u, u_a, u_b))</td>
</tr>
<tr>
<td>Elliptic</td>
<td>(u_{aa} + u_{bb} = \Phi(a, b, u, u_a, u_b))</td>
</tr>
</tbody>
</table>
We transform a PDE into canonical form by using the coordinate transformations

\[ a = a(x, y) \]
\[ b = b(x, y). \]

We can then compute the necessary derivatives using the Chain Rule. For instance,

\[ u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial u}{\partial b} \frac{\partial b}{\partial x} = u_xa + u_yb_x \]
\[ u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \frac{\partial b}{\partial y} = u_xa + u_yb_y \]

(a) Write out \( u_{xx}, u_{xy}, u_{yy} \). Be careful with the Chain Rule.

We can then substitute these formulas into the original PDE as written in Problem 7 to get

\[ \bar{A} \frac{\partial^2 u}{\partial a^2} + \bar{B} \frac{\partial^2 u}{\partial a \partial b} + \bar{C} \frac{\partial^2 u}{\partial b^2} + \bar{D} \frac{\partial u}{\partial a} + \bar{E} \frac{\partial u}{\partial b} + \bar{F} u = \bar{G} \]

by combining like terms. For instance, \( \bar{A} = Aa_x^2 + Ba_xa_y + Ca_y^2 \).

(b) Determine \( \bar{A}, \ldots, \bar{G} \).

In the case of the second canonical form of a hyperbolic equation we need \( \bar{A} = \bar{C} = 0 \). In other words,

\[ \bar{A} = Aa_x^2 + Ba_xa_y + Ca_y^2 = 0 \]
\[ \bar{C} = Ab_x^2 + Bb_xb_y + Cb_y^2 = 0. \]

If we divide through by \( a_y \) in the first and \( b_y \) in the second we get

\[ A[a_x/a_y]^2 + B[a_x/a_y] + C = 0 \]
\[ A[b_x/b_y]^2 + B[b_x/b_y] + C = 0. \]

Note that these are quadratic equations in the variable \( [a_x/a_y] \) and \( [b_x/b_y] \), which can be solved using the quadratic formula.

Thus

\[ a_x/a_y = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \]
\[ b_x/b_y = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \]

Actually, they both generate the same solutions, so we assign different signs to the two variables in order to ensure different solutions. Note that this PDE was assumed to be hyperbolic, thus the radical will produce a real number.

Finally, we ask for the ratios \( a_x/a_y \) and \( b_x/b_y \) to be constants, which will make our lives easier. This means we want any change in \( a \) to be a constant, which can be written as \( da = a_x dx + a_y dy = 0 \), or \( dy/dx = -a_x/a_y \). Similarly, this shows that \( dy/dx = -b_x/b_y \). We have thus created two ODE’s that can be used to develop the coordinate transformations needed to put a hyperbolic PDE into canonical form.

These ODE’s are given by

\[ \frac{dy}{dx} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \]
\[ \frac{dy}{dx} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \frac{B + \sqrt{B^2 - 4AC}}{2A}. \]
Let’s convert the PDE
\[ y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0, \]
for \( x > 0 \) and \( y > 0 \). First \( B^2 - 4AC = 0 - 4(y^2)(-x^2) > 0 \), and our PDE is hyperbolic. We use the information derived above to determine that
\[
\begin{align*}
\frac{dy}{dx} &= \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{0 - \sqrt{4x^2y^2}}{2y^2} = -\frac{x}{y} \\
\frac{dy}{dx} &= \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{0 + \sqrt{4x^2y^2}}{2y^2} = \frac{x}{y} 
\end{align*}
\]
(c) These ODE’s are separable. Show that the solutions to the ODE’s are \( y^2 + x^2 = c_1 \) and \( y^2 - x^2 = c_2 \).

We thus use the coordinate transformation defined by
\[
\begin{align*}
a(x, y) &= y^2 - x^2 \\
b(x, y) &= y^2 + x^2 
\end{align*}
\]
to convert our PDE into canonical form. Verify that \( \tilde{A} = \tilde{C} = 0 \).
(d) Use the transformation to show that the canonical form of the given PDE is given by
\[
\frac{\partial^2 u}{\partial a \partial b} = \frac{bu_a - au_b}{2(a^2 - b^2)}
\]
(c) Verify that
\[ 3u_{xx} + 7u_{xy} + 2u_{yy} = 0 \]
is a hyperbolic PDE, and then use the process outlined above to convert the PDE into the canonical form
\[ u_{ab} = \Phi(a, b, u, u_a, u_b). \]

Hint: You can check your transformation by using d’Alembert’s Method of factoring the operator into
\[
\left( 3 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} \right) u = 0.
\]
You should get the same transformation with both methods. In fact, the examples we have done with d’Alembert’s Method have all been hyperbolic PDE’s. Note that the method outlined above, though, works for more complicated PDE’s than what can be handled by d’Alembert’s method.