4.3.20

a) \( B = P^{-1}AP \Rightarrow B^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T = P^T A^T (P^T)^{-1} \), so \( B^T \) is similar to \( A^T \).

b) False. Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). \( A \) and \( B \) are not similar, since \( P^{-1}BP \neq A \) for any nonsingular \( P \). But \( A^2 = B^2 = I \), so \( P^{-1}B^2P = A^2 \) for all nonsingular \( P \).

c) \( B = P^{-1}AP \). Since \( A, P, \) and \( P^{-1} \) are nonsingular, so is \( B \) by Prop 3.4 p108.

d) False. See example 12 p219.

e) False. \( A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) are similar.

(Use \( P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = P^{-1} \). Observe \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in N(A) \), but \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \notin N(B) \).

So \( N(A) \neq N(B) \).

f) \( B = P^{-1}AP \) (all matrices \( n \times n \)). Then \( N(B) = N(P^{-1}AP) = N(AP) \) by prob 3.2.10c. Therefore \( \text{rank } B = n - \dim N(B) = n - \dim N(AP) = \text{rank } (AP) \).

Also \( C(AP) = C(A) \) (3.2.10d, note \( P \) nonsingular), so \( \text{rank } (AP) = \text{rank } A \). Thus \( \text{rank } B = \text{rank } A \).

5.1.6

\[
\begin{vmatrix} 1 & 8 & 9 & 8 \\ 3 & 4 & 7 & 1 \\ 7 & 2 & 1 & 5 \\ 1 & 6 & 4 \end{vmatrix}
= \begin{vmatrix} 1 & 8 & 9 & 1898 \\ 3 & 4 & 7 & 3471 \\ 7 & 2 & 1 & 7215 \\ 8 & 1 & 6 & 8164 \end{vmatrix}
= 13 \begin{vmatrix} 1 & 8 & 9 & 146 \\ 3 & 4 & 7 & 267 \\ 7 & 2 & 1 & 555 \\ 8 & 1 & 6 & 628 \end{vmatrix}
\]

Add 1000 x col 1 to col 4
100 x col 2
10 x col 3

See cor 1.8.3 p244
6.1.7 a) The inductive hypothesis $A^x = \lambda x$ is given. Now suppose $A^{n+1}x = \lambda^n x$ is true for some $n$. Multiply by $A$: $A^{n+1}x = \lambda^n A x = \lambda^n (\lambda x) = \lambda^{n+1} x$. By induction, the statement $A^n x = \lambda^n x$ is true for $n = 1, 2, 3, \ldots$

b) True: $A x = \lambda x$ \Rightarrow $(A + I)x = A x + x = \lambda x + x = (\lambda + 1)x$, so $x$ is an eigenvector of $A + I$ corresponding to the eigenvalue $\lambda + 1$.

c) True: we have $A x = \lambda x$ and $B x = \mu x$. Add to get $A x + B x = \lambda x + \mu x$, or $(A + B)x = (\lambda + \mu)x$.

d) False. Just about any $2 \times 2$ matrices $A$ and $B$ chosen at random should work as a counterexample. $A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$ has eigenvalues 3 and -3; $B = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 2 and 2. But $A + B$ has eigenvalues $2 \pm \sqrt{17}$, clearly not the sum of 3 and 2 or -3 and 2.